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Geometry and Phase Structure of Non-Relativistic Branes

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Abstract: We use the solution generating technique, the TsT transformation, to obtain new solutions in type II string theory as well as in M-theory. We explicitly work out examples starting with rotating D3 and M2 branes as well as D1-D5-p system. Among a variety of solutions, we find many of them have asymptotic Schrödinger symmetry. We also device a new method of deriving free energy of black brane systems, which is more efficient than the Euclidean action procedure. We test our method on known examples before applying it to new asymptotically Schrödinger backgrounds. We study phase structure of these backgrounds by analysing the free energy thus derived.

Keywords: AdS/CFT, Schrodinger spacetime, black hole thermodynamics.

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1. Introduction and Summary

AdS/CFT correspondence relates conformal field theories in d spacetime dimensions to superstring/M-theory compactified down to anti-de Sitter spacetime in $d+1$ dimensions[1, 2]. This correspondence has been extended to non-conformal field theories by relating it to asymptotically AdS spacetimes in the bulk. It has also been generalized to non-conformal branes[3].

Another direction in which this correspondence is extended is in the exploration of holographic duals of strongly coupled non-relativistic conformal field theories[4–11]. These non-relativistic conformal field theories have Schrödinger group symmetry[12–20]. This symmetry consists of the usual Galilean invariance, the scaling symmetry as well as the particle number symmetry[1]. There has been some activity along this direction where solution generating techniques have been used to obtain bulk geometries with asymptotic Schrödinger symmetry[10, 25–28]. A large class of asymptotically Schrödinger geometries have been obtained by applying these techniques to various brane configurations[25, 26].

Interplay between the TsT transformation and the near horizon extremal limit has been studied earlier in the case of spinning D3 brane problem[26]. It was shown that the procedure of near horizon extremal limit does not commute with the TsT transformation in case of both null TsT as well as space-like TsT transformation of spinning D3 brane solutions. In this paper, we will study TsT transformations of D-brane systems and their near horizon extremal limit. In section 2, we will write down general properties of TsT transformations and list relevant formulae which will be useful in subsequent sections. In section 3, we will re-examine the spinning D3 brane case both for null and space-like TsT [2]. We find that, our results are in agreement with earlier results in case of null TsT transformations but contrary to earlier claims we find that space-like TsT transformation does commute with the near horizon extremal limit. Later in the same section we will look at transformation of the D1-D5-p system with respect to null and space-like TsT. We find that near horizon extremal limit of this system for null TsT is independent of the parameter γ, which appears in the TsT transformation. As a result, near horizon extremal limit of D1-D5-p system before and after null TsT transformation gives rise to identical background.

We then proceed to apply this method to the rotating M2 brane solution[3] in section 4. Since T-duality is not a symmetry of M-theory, we can implement the TsT transfor-

[1] Relation of Schrödinger symmetry and the Newton-Cartan theory dates back to work of Duval et al. [21]. For other related works, see, [22–24].

[2] We will define null and space-like TsT transformations in the next section.

[3] Similar Schrödinger backgrounds from M-theory were considered earlier in [29, 30].

mation by first reducing the M2-brane solution to type IIA theory and then carrying out the TsT transformation on it. We then oxidize the resulting solution back to M-theory. There are a couple of ways of reducing the M2-brane to type IIA string theory solution. One of them gives rise to fundamental string solution whereas the other gives the D2-branes solution. Action of TsT transformation on these two solutions gives different backgrounds. In case of fundamental string solution, null TsT transformation gives rise to a background which is not asymptotically Schrödinger type geometry. The scalar $\mathcal{M}$, which parametrizes scale factor in front of light cone part of the metric as well as behavior of the dilaton, does not go to constant asymptotically. In order to get better behaved solution we work in a suitable type IIB frame, which can be obtained by doing a T-duality on type IIA background and followed by an S-duality. In this type IIB frame we carry out the TsT transformation. We then revert back to the M-theory metric in two different ways. We can either follow same steps in reverse order, *i.e.*, doing S-duality, followed by T-duality to type IIA theory and then lift to M-theory or we can directly T-dualize the background in to type IIA back ground and lift it the M-theory. These two procedures give rise to two different M-theory solutions. In the former case we end up with a geometry which has a decoupled metric piece corresponding to $\mathcal{CP}^3 \times S^1$ and the full metric asymptotically approaches $AdS_3 \times \mathcal{CP}^3 \times S^1$. In the latter case, we have decoupled $\mathcal{CP}^3$ metric and the circle is fibered over the three dimensional non-compact space. This space asymptotically becomes a circle fibration over AdS_3 with a decoupled $\mathcal{CP}^3$ metric.

In section 5, we discuss the phase structure of the TsT transformed black holes. The phase structure of the black hole spacetime can be obtained from its free energy. Therefore one needs to compute the free energy from the Euclidean path integral of black hole spacetime,

$$F = -T \ln Z_S \,, \tag{1.1}$$

where, Z_S is the partition function. For pure gravity the partition function Z_S is given by

$$Z_S(X) = \int [\mathcal{D}g]\, e^{-I_S(X)} \,, \tag{1.2}$$

where, $I_S(X)$ is the Euclidean effective action on X. In large N limit (or $G_N \to 0$, classical supergravity) it is given by[4],

$$\begin{aligned} I_S &\sim \frac{1}{G_N}\int d^{d+1}x\sqrt{g}\; R \\ &= N^2 K(X). \end{aligned} \tag{1.3}$$

[4]We are considering only gravity for simplicity.

In general, there may be several possible X's given a fixed boundary geometry M. In that case there is no particular way to pick one specific spacetime. Therefore, one has to consider all possible X's and replace (1.2) by sum over all such integral. In semi-classical limit, *i.e.* $N \to \infty$, one can write the partition function in the following way,

$$Z_S(X) = \sum_i e^{-N^2 K^{cl}(X_i^0)}\,, \tag{1.4}$$

where, X_i^0's are classical solutions of the Einstein equation with some fixed boundary geometry.

One can notice that there is a natural mechanism for a singularity or phase transition that would arise only in the large N limit. At large N, the sum will be dominated by those X_i^0 for which $F(X_i^0)$ is smallest. Therefore at a point at which $F(X_i^0) = F(X_j^0)$ for some $i \neq j$, one may jump from one branch to other branch. This signifies a phase transition in the large N theory.

To study the phase transition we need to compute the on-shell action. In this paper we develop a new method to compute the Euclidean on-shell action. In the usual way of computing on-shell action one encounters large volume divergence and one needs to regularize the action by either subtracting the contribution from background spacetime or adding counterterms to the original action. In practice, these methods are cumbersome especially when one has several matter fields. We use Wald's formula to compute entropy of the black hole which depends only on the near-horizon geometry. We derive temperature of the Euclidean black hole by computing the periodicity of Euclidean time circle. We then use these two variables to compute the on-shell action of the black hole using the following relations,

$$\begin{aligned} &S - \beta\left(\frac{\partial I}{\partial \beta}\right)_\mu + I = 0 \qquad \text{for fixed chemical potential } \mu\,, \\ &S - \beta\left(\frac{\partial I}{\partial \beta}\right)_q + I = 0 \qquad \text{for fixed charge } q\,. \end{aligned} \tag{1.5}$$

When we integrate these first order equations we encounter an integration constant. We fix this constant by demanding the free energy of the background spacetime to be zero. We describe this procedure in detail in section (5), and work out an illustrative example before applying it to the non-relativistic case. The Euclidean on-shell action derives in this manner for black hole spacetimes is then used to study phase structure. We will conclude with discussion of our results.

2. The TsT Transformation

In this section we will discuss general properties of TsT transformations, which is a solution generating technique in string theory. The basic idea of solution generating technique is to exploit the fact that the low energy supergravity theory has more symmetry than the full string theory. A solution to supergravity equations of motion can be shown to be a solution to string theory. A symmetry transformation of this solution with respect to a supergravity symmetry does not give rise to a new solution in supergravity. However, if the transformation employed is not a symmetry of full string theory then the transformed solution can be interpreted as a new solution of string theory. In case of the TsT transformations, T-dualities are performed along the isometry direction, whereas s-transformation mixes the dualized coordinate with another isometry direction. This transformation generically changes asymptotics of the resulting metric but is guaranteed to be a solution to the supergravity equations of motion. We will spell out the procedure involved in TsT transformation shortly but before that, we will briefly mention two metrics, which give rise to non-relativistic isometries in the boundary conformal field theories and appear in the TsT transformed solutions.

The holographic dual of a d spatial dimensional Galilean CFT is given by [4, 5, 7],

$$ds^2 = r^2 \left(-2dudv - r^{2(z-1)}du^2 + \sum_{i=1}^{d}(dx^i)^2 \right) + \frac{dr^2}{r^2}. \tag{2.1}$$

For $z = 1$ the spacetime becomes pure AdS_{d+3} whose dual theory corresponds to a conformal field theory in $(d+1, 1)$ spacetime dimensions.

For $z > 1$ the the bulk geometry corresponds to a dual to a non-relativistic field theory with the Galilean transformations along with scaling as a global symmetry. While the coordinate u is interpreted as a boundary time coordinate, the null direction v is taken to be compact and the quantized momentum along v is identified with the particle number of Galilean CFT. We call this spacetimes Sch^z_{d+3}, where, z is the dynamical exponent of the Schrödinger spacetime.

Let us also note that the bulk geometry corresponding to Lifshitz spacetime is given by,

$$ds^2 = r^2 \left(-r^{2(z-1)}dt^2 + \sum_{i=1}^{d}(dx^i)^2 \right) + \frac{dr^2}{r^2}, \tag{2.2}$$

where t is the boundary time coordinate of the boundary theory as well.

We will now discuss the generic form of the TsT transformed geometry. We will use the unhatted variables to denote the metric, dilaton and the second rank antisymmetric tensor before TsT transformation and hatted variables will denote them

after TsT. In case of Ramond-Ramond(RR) field strengths our convention is to use F_q to denote q-form before TsT and $\mathcal{F}_q$ to denote them after TsT. In the next section, we will apply this technique to specific examples in string theory. The TsT transformations can be applied to those metrics which have at least two isometry directions. In absence of NS-NS two form field B, it is convenient to cast the metric in the following form

$$ds^2 = (A_1 d\chi + K_1)^2 + (A_2 d\psi + A_3 d\chi + K_2)^2 + ds_8^2\,, \tag{2.3}$$

where ds_8^2 and the one-forms K_1, K_2 do not depend on the isometry directions χ and ψ. The TsT transformations then correspond to performing T-duality along ψ direction, followed by a shift along χ, *i.e.*, $\chi \to \chi - \gamma\psi$, and then performing T-duality back along the ψ direction. Notice that under the Buscher duality transformations[32], pure metric background can give rise to non-trivial dilaton and anti-symmetric tensor field background. For example, if the original metric is of the form (2.3) then the end result of the TsT transformation is

$$\begin{aligned} d\hat{s}^2 &= \mathcal{M}(A_1 d\chi + K_1)^2 + \mathcal{M}(A_2 d\psi + A_3 d\chi + K_2)^2 + ds_8^2\,, \\ e^{2\hat{\Phi}} &= \mathcal{M} e^{2\Phi}\,, \quad \mathcal{M} = (1 + \gamma^2 A_1^2 A_2^2)^{-1}\,, \\ \hat{B} &= -\gamma \mathcal{M}\, A_1 A_2\, (A_1 d\chi + K_1) \wedge (A_2 d\psi + A_3 d\chi + K_2)\,. \end{aligned} \tag{2.4}$$

In this simplified situation, where $B = 0$ and only p-form RR field strength is non-vanishing, we can easily find the effect of TsT on the RR field. Suppose this p-form field strength has components along both ψ and χ direction, then after TsT transformation we get $(p-2)$-form field strength

$$\mathcal{F}_{p-2} = \gamma \iota_\chi \iota_\psi F_p\,, \tag{2.5}$$

in addition to the original p-form field strength. As a result, the effective p-form field strength is then given by

$$F_p = \mathcal{F}_p + \mathcal{F}_{p-2} \wedge \hat{B}\,. \tag{2.6}$$

In general, original background can have non-trivial dilaton as well as antisymmetric field strength. To find the TsT transformed form of the metric and the NS-NS field strength it is useful to define [31],

$$E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu}. \tag{2.7}$$

Under TsT transformation, $E_{\mu\nu}$ transforms as,

$$\hat{E}_{\mu\nu} = \mathcal{M}\left[E_{\mu\nu} + \gamma\left[\det\begin{pmatrix} E_{\psi\chi} & E_{\psi\nu} \\ E_{\mu\chi} & E_{\mu\nu}\end{pmatrix} - \det\begin{pmatrix} E_{\chi\psi} & E_{\chi\nu} \\ E_{\mu\psi} & E_{\mu\nu}\end{pmatrix}\right] + \gamma^2 \det\begin{pmatrix} E_{\psi\psi} & E_{\psi\chi} & E_{\psi\nu} \\ E_{\chi\psi} & E_{\chi\chi} & E_{\chi\nu} \\ E_{\mu\psi} & E_{\mu\chi} & E_{\mu\nu}\end{pmatrix}\right] \tag{2.8}$$

where,

$$\mathcal{M} = \left[1 + \gamma(E_{\psi\chi} - E_{\chi\psi}) + \gamma^2 \det \begin{pmatrix} E_{\psi\psi} & E_{\psi\chi} \\ E_{\chi\psi} & E_{\chi\chi} \end{pmatrix}\right]^{-1} . \tag{2.9}$$

We can now read out the transformed metric and NS-NS antisymmetric field from the equation (2.8) by isolating symmetric and antisymmetric parts of $\hat{E}$, namely,

$$\begin{aligned} \hat{g}_{\mu\nu} &= \mathrm{Sym}[\hat{E}_{\mu\nu}] \, , \\ \hat{B}_{\mu\nu} &= \mathrm{AntiSym}[\hat{E}_{\mu\nu}] \, . \end{aligned} \tag{2.10}$$

The dilaton transforms as,

$$e^{2\hat{\Phi}} = \mathcal{M} \; e^{2\Phi}. \tag{2.11}$$

The TsT transformation also acts on the RR field strengths. For general backgrounds, RR fields and the field strengths after TsT are related to those before TsT

$$\begin{aligned} \sum_q \hat{C}_q \wedge \exp(\hat{B}) &= \sum_q C_q \wedge \exp(B) + \gamma \iota_\chi \iota_\psi \sum_q C_q \wedge \exp(B) \, , \\ \sum_q \mathcal{F}_q \wedge \exp(\hat{B}) &= \sum_q F_q \wedge \exp(B) + \gamma \iota_\chi \iota_\psi \sum_q F_q \wedge \exp(B) \, , \end{aligned} \tag{2.12}$$

where B is the Neveu-Schwarz sector antisymmetric tensor before TsT transformation and $\hat{B}$ is that after TsT. In the next two sections, we will apply these general TsT transformation rules to different string theory and M-theory geometries to generate deformed solutions.

3. *TsT* Transformation of String Geometry

In this section we discuss the effect of TsT transformation of Dp brane geometry and study how their asymptotic and near horizon geometries change under this transformation. In particular we consider two different D brane configurations, rotating $D3$ brane geometry and $D1$-$D5$-p system. Both arise as solutions of type IIB string theory. We perform null as well as space-like TsT transformations in both the cases and discuss the behavior of geometries asymptotically as well as in the near horizon limit. We also study the commutativity of extremal near horizon limit and TsT transformation in both the cases.

3.1 Rotating $D3$ Brane

We start with a review on the non-relativistic extension of the rotating non-extremal $D3$-brane solution. Let us consider $D3$ branes rotating along three isometry directions

of transverse S^5 space with equal angular momenta in all three planes. From five dimensional point of view (compactifying over S^5) this system can be viewed as a charged AdS_5 black brane with $U(1)^3$ symmetry. The boundary theory corresponds to a conformal field theory with three equal global $U(1)$ charges. This system has been studied in details in [26]. In most of the cases our results are in agreement with theirs, however, there are some differences in case of space-like TsT case.

We will be working in ten dimensional set-up and we will analyze effect of both Null TsT and space-like TsT transformations. Since TsT transformation is a symmetry of supergravity, the non-relativistic geometries resulting out of these transformations are solutions of type IIB theory. In the end we will take extremal near horizon limit of both TsT transformed solutions. As mentioned in the introduction, our results agree with earlier results in the literature about non-commutativity of null TsT transformation and the near horizon extremal limit. In case of space-like TsT, however, we find that it commutes with the near horizon extremal limit. The non-extremal metric[5] in the decoupling limit [1] of the rotating $D3$ brane geometry is given by,

$$
\begin{aligned}
ds^2 &= \frac{r^2}{l^2}\left(-f dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{l^2}{r^2}\frac{dr^2}{f} \\
&\quad + l^2\left(d\alpha^2 + \sin^2\alpha d\beta^2 + \mu_1^2(d\xi_1 + A)^2 + \mu_2^2(d\xi_2 + A)^2 + \mu_3^2(d\xi_3 + A)^2\right) , \qquad (3.1)\\
F_5 &= (1 + *)\left[\left(-\frac{4r^3}{l^4}dt \wedge dr + \frac{Q}{l^2}\, d\left(\sum_{i=1}^{3}\mu_i^2 d\xi_i\right)\right) \wedge dx \wedge dy \wedge dz\right] , \qquad (3.2)
\end{aligned}
$$

where the angular functions μ_i are parameterized as,

$$
\mu_1 = \cos\alpha\,, \qquad \mu_2 = \sin\alpha\cos\beta\,, \qquad \mu_3 = \sin\alpha\sin\beta\,, \tag{3.3}
$$

and the metric functions are as follows,

$$
f(r) = \left(1 - \frac{r_0^2}{r^2}\right)\left(1 + \frac{r_0^2}{r^2} - \frac{Q^2}{r_0^2 r^4}\right) , \qquad \mathcal{A} = \mathcal{A}_t\, dt = \frac{Q}{l^2}\left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) dt\,. \tag{3.4}
$$

In what follows we will consider three angular momenta to be equal and denote them by Q.

As mentioned in the previous section we will need two isometry directions to carry out the TsT transformation. We will use one of the D3-brane world volume direction as an isometry direction. Second isometry direction that we will use will be from S^5 direction. To extract that it is suitable to write the S^5 metric as a U(1) fibration over $\mathbb{CP}^2$,

$$
ds^2_{S^5} = (d\psi + \mathcal{P} + \mathcal{A})^2 + ds^2_{\mathrm{CP}^2}\,, \tag{3.5}
$$

[5] We follow the notation of [26].

where the one form $\mathcal{A}$ is defined in (3.4) and,

$$\mathcal{P} = \frac{1}{3}\left(\mathrm{d}\chi_1 + \mathrm{d}\chi_2\right) - \sin^2\alpha\left(\mathrm{d}\chi_2\sin^2\beta + \mathrm{d}\chi_1\cos^2\beta\right), \tag{3.6}$$

$$\psi = \frac{1}{3}(\phi_1 + \phi_2 + \phi_3), \quad \chi_1 = \phi_1 - \phi_2, \quad \chi_2 = \phi_1 - \phi_3 \,. \tag{3.7}$$

Note that curvature of the one form $\mathcal{P}$ is proportional to the Kähler form $\omega_{\mathbb{CP}^2}$ on $\mathbb{CP}^2$, namely, $-\frac{l^2}{2}d\mathcal{P} = \omega_{\mathbb{CP}^2}$ and the metric of $\mathbb{CP}^2$ can be expressed as,

$$\begin{aligned} ds^2_{\mathrm{CP}^2} = {} & d\alpha^2 + \sin^2\alpha d\beta^2 \\ & + \sin^2\alpha\cos^2\alpha(\cos^2\beta d\chi_1 + \sin^2\beta d\chi_2)^2 \\ & + \sin^2\alpha\sin^2\beta\cos^2\beta(d\chi_1 - d\chi_2)^2 \,. \end{aligned} \tag{3.8}$$

Thus, the rotating D3 brane solution that we are looking at has all three angular momenta equal. In the next sub-sections we will study the non-relativistic extension of this geometry.

3.1.1 Null TsT Transformation

We will start with the null TsT transformation. Before performing the transformation, we first need to define light-cone co-ordinates,

$$x^{\pm} = \frac{1}{2}(t \pm y). \tag{3.9}$$

The vielbeins e^i, which are derived from the original metric, are given by,

$$\begin{aligned} e^0 &= \frac{2r\sqrt{f}}{\sqrt{1-f}}dx^+, \quad e^1 = \frac{r\left[(1-f)dx^- - (1+f)dx^+\right]}{l\sqrt{1-f}}, \quad e^2 = \frac{r}{l}dx\,, \quad e^3 = \frac{r}{l}dz, \\ e^4 &= \frac{dr}{r\sqrt{1-f}}, \quad e^5 = l d\alpha\,, \quad e^6 = l\sin\alpha d\beta\,, \quad e^7 = \frac{l}{2}\sin 2\alpha(\cos^2\beta d\chi_1 + \sin^2\beta d\chi_2)\,, \\ e^8 &= l\sin\alpha\sin\beta\cos\beta(d\xi_1 - d\xi_2), \quad e^9 = l(d\psi + \mathcal{P} + \mathcal{A}). \end{aligned} \tag{3.10}$$

To connect up to the notation of sec.2, we will use x^- as χ direction in the TsT transformation described there. In the light-cone coordinates, different functions appearing in the metric (2.3) take the following form:

$$\begin{aligned} A_1 &= \frac{r}{l}\sqrt{1-f}, \qquad A_2 = l\,, \quad A_3 = \mathcal{A}_t\,, \qquad K_1 = -\frac{r}{l}\frac{(1+f)}{\sqrt{1-f}}dx^+\,, \\ K_2 &= \frac{1}{3}(d\chi_1 + d\chi_2) - \sin^2\alpha\left(d\chi_1\cos^2\beta + d\chi_2\sin^2\beta\right) + \mathcal{A}_t dx^+\,. \end{aligned} \tag{3.11}$$

Notice that the NS-NS two form B is zero for the D3 brane solution (3.1). Following the general TsT transformation procedure outlined in sec.2, we write down the new solution obtained from eq.(3.1). The TsT procedure is implemented after expressing (3.1) in terms of the lightcone coordinates, $x^{\pm}$. The new background, which also solves type IIB equations of motion, is

$$\begin{aligned}
d\hat{s}^2 &= \mathcal{M}(A_1 dx^- + K_1)^2 + \mathcal{M}(A_2 d\psi + A_3 dx^- + K_2)^2 + ds_8^2\,, \\
e^{2\hat{\Phi}} &= \mathcal{M} = \left(1+\gamma^2 r^2(1-f)\right)^{-1}\,, \quad \hat{C}_2 = -\gamma l^2 \mathcal{A}_t\, \omega_{\mathbb{CP}^2}\,, \\
\hat{B} &= -\gamma r \mathcal{M}\sqrt{1-f}(A_1 dx^- + K_1)\wedge(A_2 d\psi + A_3 dx^- + K_2)\,, \\
\mathcal{F}_5 &= F_5 + \hat{H}_3\wedge\hat{C}_2 = (1+*)\,G_5 \\
&= (1+*)\Big[-\frac{4}{l}\,e^0\wedge e^1\wedge e^2\wedge e^3\wedge e^4 + \frac{2Q}{lr^3\sqrt{1-f}}\,(\sqrt{f}e^0+e^1)\wedge e^2\wedge e^3\wedge\omega_{\mathbb{CP}^2}\Big]\,,
\end{aligned} \tag{3.12}$$

The Poincare dual is defined using $*$ is defined with respect to this TsT transformed metric. We can also write the deformed geometry (3.12) in terms of original (t,y) coordinates as,

$$\begin{aligned}
d\hat{s}^2 &= \frac{r^2}{l^2}\mathcal{M}\left(-f dt^2 + dx^2 - \gamma^2 r^2 f (dt+dy)^2\right) + \frac{r^2}{l^2}\left(dx^2+dz^2\right) + \frac{l^2}{r^2}\frac{dr^2}{f} \\
&\quad + l^2\left(\mathcal{M}\,(d\psi+\mathcal{P}+\mathcal{A})^2 + ds^2_{\mathbb{CP}^2}\right)\,, \\
e^{2\hat{\Phi}} &= \mathcal{M}\,, \quad \hat{B} = \gamma r^2 \mathcal{M}\,(f dt + dy)\wedge(d\psi+\mathcal{P}+\mathcal{A})\,, \quad \hat{C}_2 = -\gamma l^2 \mathcal{A}_t\,\omega_{\mathbb{CP}^2}\,, \\
\mathcal{F}_5 &= (1+*)\,G_5 = (1+*)\left[\left(-\frac{4r^3}{l^4}dt\wedge dr - \frac{2Q}{l^4}\,\omega_{\mathbb{CP}^2}\right)\wedge dx\wedge dy\wedge dz\right].
\end{aligned} \tag{3.13}$$

Let us now look at the asymptotic geometry $(r\to\infty)$ of the above solution. For this we will revert back to the light cone variables,

$$\begin{aligned}
d\hat{s}^2 &\to \frac{r^2}{l^2}\left(-4\gamma^2 r^2 (dx^+)^2 - 4dx^+dx^- + dx^2 + dy^2\right) \\
&\quad + \frac{l^2}{r^2}\frac{dr^2}{r^2} + l^2\left((d\psi+\mathcal{P}+\mathcal{A}_\infty)^2 + ds^2_{\mathbb{CP}^2}\right)\,, \\
\exp(2\hat{\Phi}) &\to 1, \quad \hat{B}\to \gamma\, r^2\, dx^+\wedge(d\psi+\mathcal{P}+\mathcal{A}_\infty)\,, \\
\hat{C}_2 &\to -\gamma\, l^2\, \mathcal{A}_{t\infty}\,\omega_{\mathbb{CP}^2}.
\end{aligned} \tag{3.14}$$

Defining rescaled light-cone coordinates as,

$$u = 2\gamma l x^+, \qquad v = \frac{1}{\gamma l}x^-\,, \tag{3.15}$$

it is easy to see that the above metric takes the form (2.1) at the boundary $r\to\infty$ with dynamical exponent $z=2$.

Extremal Near-Horizon Limit

We are interested in studying the near horizon extremal limit of this deformed geometry. As is well known, that the undeformed geometry (3.1) has a AdS_2 factor in its extremal near-horizon limit. The extremal limit is arrived at by taking, $Q = \sqrt{2} r_0^3$. Taking near-horizon limit of the extremal geometry is little subtle and usually one needs to do an appropriate scaling of coordinates to get to the near horizon geometry. We will consider the following scaling limit,

$$\rho = \frac{r - r_0}{\beta}, \qquad \tau = \beta t, \quad \text{with} \quad \beta \to 0. \tag{3.16}$$

This gives the near-horizon extremal geometry as,

$$\begin{aligned}
d\hat{s}^2 &= \tfrac{r_0^2}{l^2}\left(-\tfrac{12\rho^2}{r_0^2}dt^2 + \mathcal{M}_0\, dy^2\right) + \tfrac{r_0^2}{l^2}\left(dx^2 + dz^2\right) + \tfrac{l^2}{12\rho^2}d\rho^2 \\
&\qquad\qquad + l^2\left(\mathcal{M}_0\left(d\psi + P + \tfrac{2\sqrt{2}\rho}{l^2}dt\right)^2 + ds^2_{\mathbb{CP}^2}\right), \\
e^{2\hat{\Phi}} &= \mathcal{M}_0\,, \quad \hat{B} = \gamma r_0^2\, \mathcal{M}_0\, dy \wedge \left(d\psi + P + \tfrac{2\sqrt{2}\rho}{l^2}dt\right), \quad \hat{C}_2 = 0\,, \\
\mathcal{F}_5 &= (1 + *)\left[\left(-\tfrac{4r_0^3}{l^4}dt \wedge d\rho - \tfrac{2\sqrt{2}r_0^3}{l^4}\,\omega_{\mathbb{CP}^2}\right) \wedge dx \wedge dy \wedge dz\right],
\end{aligned} \tag{3.17}$$

where, $\mathcal{M}_0 = (1 + \gamma^2 r_0^2)^{-1}$.

Thus we see that, while the asymptotic geometry has Schrödinger isometry, we do get the AdS_2 factor in the near horizon extremal limit. The same geometry (3.17) can also be obtained, if we start from the extremal near horizon solution (which has an AdS_2 factor in it), perform null TsT and then take the above scaling limit (3.16). It is worth mentioning that in this case before taking the scaling limit, the near horizon geometry does not have any AdS_2 isometry, as pointed out by [26].

3.1.2 Space-Like TsT Transformation

For space-like TsT transformation we do not use the light-cone coordinates, we instead choose y as χ direction and perform the twist in y direction. The deformed geometry now looks like:

$$\begin{aligned}
d\hat{s}^2 &= \frac{r^2}{l^2}\left(-f dt^2\right) + \mathcal{M}\frac{r^2}{l^2}dy^2 + \frac{r^2}{l^2}\left(dx^2 + dz^2\right) + \frac{l^2}{r^2}\frac{dr^2}{f} \\
&\qquad + l^2\left(\mathcal{M}\left(d\psi + \mathcal{P} + \mathcal{A}\right)^2 + ds^2_{\mathbb{CP}^2}\right), \\
e^{2\hat{\Phi}} &= \mathcal{M}\,, \quad \hat{B} = -\gamma r^2 \mathcal{M}\, dy \wedge \left(d\psi + \mathcal{P} + \mathcal{A}\right), \quad \hat{C}_2 = -\gamma l^2 \mathcal{A}_t\, \omega_{\mathbb{CP}^2}\,, \\
\mathcal{F}_5 &= (1 + *)\, G_5 = (1 + *)\left[\left(-\frac{4r^3}{l^4}dt \wedge dr - \frac{2Q}{l^4}\,\omega_{\mathbb{CP}^2}\right) \wedge dx \wedge dy \wedge dz\right],
\end{aligned} \tag{3.18}$$

where, $\mathcal{M} = (1 + \gamma^2 r^2)^{-1}$.

The near-horizon extremal limit gives similar result as above, only with an opposite sign for the two-form field $\hat{B}$. It also solves the equations of motion as expected. For both cases, we see that the near-horizon extremal geometry does depend on the shift parameter γ. We will see that the situation is different in the $D1$-$D5$-p system that we will study in the next section. Starting with extremal near-horizon geometry and a space-like TsT and then scaling gives back the above geometry.

Thus we see that the extremal near-horizon limit of TsT transformed geometry and the TsT transform of near horizon extremal geometry are *not* same. We summarize our observations in the following (non)commutative diagram.

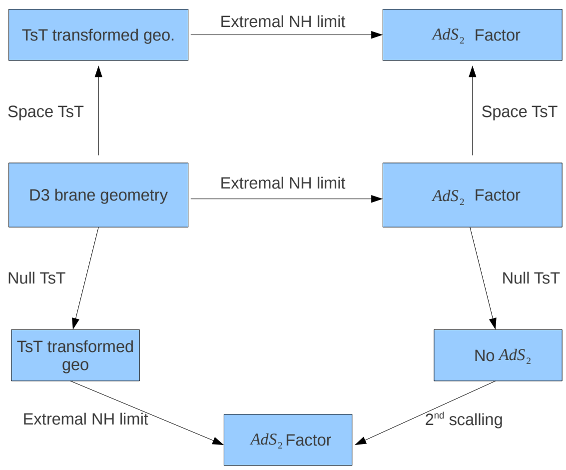

Figure 1: TsT transformation and near-horizon geometry.

3.2 D1-D5-p System

We will now look at another solution of type IIB theory. This is given by the D1-D5 brane system with momentum p, where Q_1 number of D1 branes wrap the compact direction x_5, Q_5 D5-brane wraps x_5 and a compact four manifold M_4 and the momentum n is along the circle x_5. In the string frame, the metric, the dilaton and the RR 3-form

field strength F_3 corresponding to this solution are given by[33],

$$
\begin{aligned}
ds^2_{\rm str} &= \frac{1}{\sqrt{H_1 H_2}}\Bigg[-dt^2 + dx_5^2 + \frac{c_3}{r^2}(\coth\sigma dt + dx_5)^2 + H_1 dM_4^2\Bigg] \\
&\quad + \sqrt{H_1 H_2}\Bigg[f^{-1}dr^2 + r^2 d\Omega_3^2\Bigg], \\
e^{-2\phi} &= \left(1+\frac{r_0^2 \sinh^2\gamma}{r^2}\right)\left(1+\frac{r_0^2\sinh^2\alpha}{r^2}\right)^{-1}, \\
F_3 &= \frac{2\sqrt{c_1(c_1+r_0^2)}}{r^3 H_1^2} dt\wedge dx_5 \wedge dr + \frac{2\sqrt{c_2(c_2+r_0^2)}}{r^3}\epsilon_3, \qquad (3.19)
\end{aligned}
$$

where ϵ_3 is the volume element of 3 sphere. The harmonic functions appearing in the metric are,

$$
H_1 = 1 + \frac{c_1}{r^2}, \qquad H_2 = 1+\frac{c_2}{r^2} \qquad f = 1 - \frac{r_0^2}{r^2}, \qquad (3.20)
$$

and the parameters c_i are as follows,

$$
c_1 = r_0^2 \sinh^2\alpha, \qquad c_2 = r_0^2\sinh^2\gamma, \qquad c_3 = r_0^2 \sinh^2\sigma\,. \qquad (3.21)
$$

The corresponding brane charges and the momentum are expressed in terms of c_i, the volume of M_4, V, the radius of x_5 circle, R and the string coupling g:

$$
\begin{aligned}
Q_1 &= \frac{V}{4\pi^2 g}\int *F_3 = \frac{V}{2g}2\sqrt{c_1(c_1+r_0^2)}\,, \\
Q_5 &= \frac{1}{4\pi^2 g}\int F_3 = \frac{1}{2g}2\sqrt{c_2(c_2+r_0^2)}\,, \\
n &= \frac{R^2 V r_0^2}{2g^2}\sinh 2\sigma\,. \qquad (3.22)
\end{aligned}
$$

Extremal limit of this background corresponds to taking α, γ, σ to ∞ and $r_0 \to 0$ keeping c_1, c_2, c_3 fixed, *i.e.*,

$$
\lim_{\substack{r_0\to 0\\ \alpha\to\infty}} r_0^2\sinh^2\alpha \to c_1, \qquad \lim_{\substack{r_0\to 0\\ \alpha\to\infty}} r_0^2\sinh^2\gamma \to c_2, \qquad \lim_{\substack{r_0\to 0\\ \alpha\to\infty}} r_0^2\sinh^2\sigma \to c_3. \qquad (3.23)
$$

We will now determine the near horizon geometry of the extremal solution. The near-horizon extremal geometry, as expected, has a AdS_2 factor. To obtain this, we first do the following coordinate transformation

$$
r = \sqrt{c_3\lambda\tilde{\rho}}, \quad x_5 = y - t, \quad \tau = \lambda t, \qquad (3.24)
$$

and then take $\lambda \to 0$ which corresponds to the near-horizon limit. The near-horizon extremal metric becomes,

$$ds^2_{\text{str}} = \frac{\sqrt{c_1 c_2}}{4}\left(-\frac{4c_3}{c_1 c_2}\tilde{\rho}^2 d\tau^2 + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2}\right) + \frac{c_3}{\sqrt{c_1 c_2}}(dy - \tilde{\rho} d\tau)^2 + \sqrt{\frac{c_1}{c_2}} dM_4^2 + \sqrt{c_1 c_2} d\Omega_3^2 \,. \tag{3.25}$$

Changing variable $\rho = 2\tilde{\rho}\sqrt{\frac{c_3}{c_1 c_2}}$ we get,

$$ds^2_{\text{str}} = \frac{\sqrt{c_1 c_2}}{4}\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho^2}\right) + \frac{c_3}{\sqrt{c_1 c_2}}(dy - \frac{1}{2}\frac{c_1 c_2}{c_3}\rho d\tau)^2 + \sqrt{\frac{c_1}{c_2}} dM_4^2 + \sqrt{c_1 c_2} d\Omega_3^2 \,. \tag{3.26}$$

Thus, we get the AdS_2 factor with momentum along the y direction. The three form field strength in this limit is given by,

$$F_3 = \frac{1}{2}\sqrt{\frac{c_2 c_3}{c_1}} d\tau \wedge dy \wedge d\rho + \frac{2c_2}{\rho^3}\epsilon_3. \tag{3.27}$$

Next, we will apply both, null and space-like TsT transformations on this geometry and find the deformed solutions.

3.2.1 Null TsT Transformation

We will work with full ten dimensional geometry (3.19). We start with the null TsT transformation, it is then convenient to define the null coordinates as,

$$x^{\pm} = \frac{1}{2}(t \pm x_5). \tag{3.28}$$

Now, in terms of these new coordinates, we can rewrite the D1-D5 metric as,

$$ds^2 = (A_1 dx^- + K_1)^2 + (A_2 d\psi + A_3 dx^- + K_2)^2 + ds_8^2, \tag{3.29}$$

where,

$$\begin{aligned}
A_1 &= \frac{\sqrt{c_3}(\coth\sigma - 1)}{r(H_1 H_2)^{\frac{1}{4}}}, \quad A_2 = r(H_1 H_2)^{\frac{1}{4}}, \quad A_3 = 0, \\
K_1 &= -\frac{2r^2 - c_3(\coth^2\sigma - 1)}{r(H_1 H_2)^{\frac{1}{4}}\sqrt{c_3}(\coth\sigma - 1)} dx^+, \quad K_2 = r(H_1 H_2)^{\frac{1}{4}}\frac{1}{2}\cos\theta d\phi,
\end{aligned} \tag{3.30}$$

and all other components of the metric are written in ds_8^2. The ds_8^2 does not play any role in what follows. The S^3 metric is written as a Hopf fibration over S^2. While the fibered coordinate is written explicitly, S^2 metric is part of ds_8^2. More concretely,

$$r^2\sqrt{H_1 H_2} d\Omega_3^2 = (A_2 d\psi + A_3 dx^- + K_2)^2 + \frac{A_2}{4}(d\theta^2 + \sin^2\theta d\phi^2) \,. \tag{3.31}$$

As far as the NS-NS two form field B is concerned the situation is similar to the D3-brane case, namely the B field is vanishing in this background. Thus, we can readily write down the TsT transformed geometry as (2.4), where,

$$\mathcal{M} = (1+\gamma^2 A_1^2 A_2^2)^{-1} = (1+\gamma^2 c_3(\coth\sigma - 1)^2)^{-1}. \tag{3.32}$$

Again the 3-form field strength F_3 does not change under this transformation.

We can rewrite the above solution in terms of original coordinates as,

$$\begin{aligned} d\hat{s}^2_{\rm str} &= \frac{\mathcal{M}}{\sqrt{H_1H_2}}\Big[-dt^2 + dx_5^2 + \frac{c_3}{r^2}(\coth\sigma dt + dx_5)^2 + \gamma^2(c_3(\coth^2\sigma - 1) - r^2)(dt+dx_5)^2\Big] \\ &+ \sqrt{\frac{H_1}{H_2}} dM_4^2 + \sqrt{H_1H_2}\frac{dr^2}{f} + \mathcal{M}(A_2 d\psi + A_3 dx^- + K_2)^2 + \frac{B}{4}(d\theta^2 + \sin^2\theta d\phi^2) \\ e^{2\hat{\Phi}} &= \mathcal{M} \\ \hat{B} &= -\gamma\mathcal{M}\, A_2\,(c_3(\coth\sigma - 1) - r^2)(\coth\sigma dt + dx_5)\wedge(A_2 d\psi + A_3 d\chi + K_2)\,, \end{aligned} \tag{3.33}$$

and F_3 does not change its form under this transformation[6].

Extremal Near-Horizon Limit

We want to study the extremal near horizon geometry of this non-relativistic geometry. The limit is defined in (3.23) and (3.24). In this limit, $\tanh\sigma \to 1$ and we can easily check that the geometry is same as the one given in (3.26); in particular the NS-NS two form field B vanishes. This fact has an interesting implication on the entropy of the non-relativistic black holes. Since extremal near horizon geometry is identical for both original and TsT transformed case implies the entropy of the non-relativistic black hole is identical to that of the relativistic one even after taking higher derivative corrections into account.

3.2.2 Space-Like TsT Transformation

Let us now look at the effect of space-like TsT transformations on the $D1$-$D5$-p system. In this case, we will choose the two symmetry directions as, x_5 and ψ. We again re-write the metric as,

$$ds^2 = (A_1 dx_5 + K_1)^2 + (A_2 d\psi + A_3 dx^- + K_2)^2 + d\tilde{s}_8^2, \tag{3.34}$$

where,

$$\begin{aligned} A_1 &= \frac{(1+\frac{c_3}{r^2})^{\frac12}}{(H_1H_2)^{\frac14}}, \quad A_2 = r(H_1H_2)^{\frac14}, \quad A_3 = 0\,, \\ K_1 &= \frac{c_3\coth\sigma}{r^2(H_1H_2)^{\frac14}(1+\frac{c_3}{r^2})^{\frac12}}dt, \quad K_2 = r(H_1H_2)^{\frac14}\frac12\cos\theta d\phi, \end{aligned} \tag{3.35}$$

[6] In [25] the authors found the geometry for $n=0$ case.

Now following the similar procedure as described above we can write the space-like TsT transformed geometry, and it looks like (2.4), with χ as x_5 and metric functions as given in (3.35). $\mathcal{M}$ takes the following form,

$$\mathcal{M} = (1 + \gamma^2 r^2 (1 + \frac{c_3}{r^2}))^{-1}, \tag{3.36}$$

and the explicit form of the background is,

$$\begin{aligned}
d\hat{s}^2_{\text{str}} &= \frac{\mathcal{M}}{\sqrt{H_1 H_2}} \Big[- dt^2 + dx_5^2 + \frac{c_3}{r^2}(\coth\sigma dt + dx_5)^2 + \gamma^2(c_3(\coth^2\sigma - 1) - r^2)dt^2 \Big] \\
&\quad + \sqrt{\frac{H_1}{H_2}} dM_4^2 + \sqrt{H_1 H_2}\frac{dr^2}{f} + \mathcal{M}(A_2 d\psi + A_3 dx^- + K_2)^2 + \frac{A_2}{4}(d\theta^2 + \sin^2\theta d\phi^2)\,, \\
e^{2\hat{\Phi}} &= \mathcal{M}\,, \\
\hat{B} &= -\gamma\mathcal{M}\, A_1 A_2\, dx_5 \wedge (A_2 d\psi + A_3 d\chi + K_2)\,.
\end{aligned} \tag{3.37}$$

We can again study the extremal near horizon limit of this geometry and we find that it is not independent of the shift parameter γ, quite unlike the null TsT transformed geometry. In the extremal limit, $\mathcal{M} \to \mathcal{M}_0 = (1 + \gamma^2 c_3)^{-1}$, and hence the overall γ dependence does not drop out. The two form NS-NS field is also non trivial in this limit.

3.2.3 *TsT* of Non-extremal $D1$-$D5$ System: BTZ Black Hole and CFT_1

In this section we consider the decoupling limit of non-extremal $D1$-$D5$ system which is $BTZ \times S^3 \times T^4$. The boundary of this AdS black hole is two dimensional. We start with the metric (3.19) and consider the following decoupling limit[7]

$$\begin{aligned}
\alpha' \to 0, \quad r \to 0, \quad r_0 \to 0\,, \\
\alpha, \gamma, \sigma \to \infty
\end{aligned} \tag{3.38}$$

with

$$U = \frac{r}{\alpha'}, \quad U_0 = \frac{r_0}{\alpha'} = \text{fixed}, \tag{3.39}$$

In this limit the metric (3.19) becomes,

$$\begin{aligned}
ds^2 = \alpha' \Bigg[\frac{U^2}{l^2}\left(-dt^2 + dx_5^2\right) + \frac{U_0^2}{l^2}\left(\cosh\sigma dt + \sinh\sigma dx_5\right)^2 + \frac{dU^2}{U^2 - U_0^2} \\
+ l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{vQ_5}}(dx_6^2 + \cdots + dx_9^2) \Bigg]\,.
\end{aligned} \tag{3.40}$$

[7]See [34] for detailed discussion

After performing the following coordinate redefinition,

$$u^2 = \frac{(U^2 + U_0^2 \sinh^2\sigma)}{l^2}, \quad \tau = lt, \quad r_+ = \frac{U_0 \cosh\sigma}{l}, \quad r_- = \frac{U_0 \sinh\sigma}{l}, \tag{3.41}$$

the BTZ black hole metric becomes,

$$\begin{aligned} ds^2 = \alpha' \Bigg[-\frac{u^2}{l^2} f d\tau^2 + u^2 \left(dx_5 + \frac{r_+ r_-}{u^2 l} d\tau \right)^2 + \frac{l^2 du^2}{u^2 f} \\ + l^2 d\Omega_3^2 + \sqrt{\frac{Q_1}{vQ_5}} (dx_6^2 + \cdots + dx_9^2) \Bigg], \end{aligned} \tag{3.42}$$

where different quantities appearing in the metric are,

$$\begin{aligned} v &= \frac{V_4}{16\pi\alpha'^2} = \text{fixed}, \quad l = (g_6^2 Q_1 Q_5)^{1/4}, \quad g_6 = \frac{g_s}{\sqrt{v}}, \\ f &= \left(1 - \frac{r_+}{u^2}\right)\left(1 - \frac{r_-}{u^2}\right), \end{aligned} \tag{3.43}$$

where, V_4 is the volume of T^4 and g_s is the string coupling. We will carry out only null TsT transformation on the BTZ black hole solution. As usual, we will again define light-cone coordinates as $x^\pm = \frac{1}{2}(t \pm l\, x_5)$ and follow the procedure outlined earlier. The final form of the TsT transformed geometry is,

$$\begin{aligned} d\hat{s}^2 &= \mathcal{M}(A_1 dx^- + K_1 dx^+)^2 + \mathcal{M}\left(A_2 d\psi + \frac{1}{2}\cos\theta d\phi\right)^2 - A_4 (dx^+)^2 \\ &\quad + \frac{l^2}{4}\left(d\theta + \sin^2\theta d\phi\right)^2 + \sqrt{\frac{Q_1}{vQ_5}}(dx_6^2 + \cdots + dx_9^2), \\ A_1 &= \frac{r_+ - r_-}{l}, \quad A_2 = l, \quad K_1 = \frac{r_+^2 + r_-^2 - 2r^2}{l(r_+ - r_-)}, \\ A_4 &= 4\frac{(r^2 - r_-^2)(r^2 - r_+^2)}{l^2 (r_+ - r_-)^2}, \quad \mathcal{M} = (1 + \gamma^2 (r_+ - r_-)^2)^{-1}. \end{aligned} \tag{3.44}$$

This metric has asymptotic Schrödinger symmetry. This can be seen by carrying out large r asymptotic expansion of the above geometry and in particular, the asymptotic metric in light-cone coordinates has the form,

$$\begin{aligned} d\hat{s}^2 = \mathcal{M}\left[-\frac{4\gamma^2 r^4}{l^2}(dx^+)^2 - \frac{2r^2}{l^2} dx^+ dx^- \right] + \mathcal{M}(d\psi + \frac{1}{2}\cos\theta d\phi)^2 \\ + \frac{l^2}{4}\left(d\theta + \sin^2\theta d\phi\right)^2 + \frac{l^2 dr^2}{r^2}. \end{aligned} \tag{3.45}$$

It is evident that the asymptotic Schrödinger background has the dynamical exponent $z = 2$.

It is worth pointing out at this point that the boundary field theory in this case is one dimensional. In other words, this geometry is dual to quantum mechanical system. Given the fact that the bulk geometry asymptotes to Schrödinger spacetime with dynamical exponent $z = 2$ implies the boundary theory is conformal quantum mechanics. This is because for $z = 2$, Schrödinger spacetimes have additional symmetry corresponding to special conformal transformations. Recall that conformal quantum mechanics plays pivotal role in Sen's quantum entropy function[35]. It would be interesting to see if a similar technique could be extended to non-relativistic black holes.

4. *TsT* Transformation of Rotating M2 Brane

In this section we will look at the rotating M2 brane solutions in the 11-dimensional supergravity (low energy limit of M-theory). These M2 branes can have four angular momenta in the transverse plane and they source the three form potential in the 11 dimension. Like in the case of D3 brane, we will start with a rotating M2 brane solution with all angular momenta equal. This background in the decoupling limit takes the form,

$$\begin{aligned} ds^2 &= -\frac{f(\rho)}{H^2}dt^2 + H^2(f(\rho)d\rho^2 + \rho^2 d\Omega_{2,k}^2) + g^{-2}\sum_{i=1}^{4}(d\mu_i^2 + \mu_i^2(d\phi_i + \mathcal{A}_i)^2, \\ C_3^M &= -8g^3\rho^3 H^3 dt \wedge d\Omega_{2,k}^2 + 2\sqrt{q(\mu + kq)}\sum_i \mu_i^2 d\phi_i \wedge d\Omega_{2,k}^2 \end{aligned} \tag{4.1}$$

where,

$$f = k - \frac{\mu}{r} + 4g^2 r^2 H^4, \quad H = 1 + \frac{q}{r}\,. \tag{4.2}$$

We will now do the following coordinate transformation,

$$r = \rho + q, \qquad m = \mu + 2qk, \qquad Q = \sqrt{q(\mu + kq)}\,. \tag{4.3}$$

From now on we will work in $k = 0$ limit, which implies, $d\Omega_{2,k}^2 = dy_1^2 + dy_2^2$. We also redefine the $M2$ brane worldvolume coordinates, $\vec{y} = 2g\vec{x}$, and also redefine the coupling constant as $g = \frac{\hat{g}}{2}$ so that we can bring the boundary metric in the form $r^2(-dt^2 + d\vec{x}^2) + M_7$. The metric and the three form field strength expressed in terms

of these variables become,

$$
\begin{aligned}
ds^2 &= -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + \frac{r^2}{\hat{g}^2}(dx_1^2 + dx_2^2) + 4\hat{g}^2 \sum_{i=1}^{4} \left(d\mu_i^2 + \mu_i^2(d\phi_i + \mathcal{A})^2\right), \\
C_3^M &= -\hat{g}r^3 dt \wedge dx_1 \wedge dx_2 + \frac{2Q}{\hat{g}^2} \sum_{i=1}^{4} \mu_i^2 d\phi_i \wedge dx_1 \wedge dx_2 \,, \qquad (4.4)
\end{aligned}
$$

and

$$
\mathcal{A} = \frac{Q}{r\hat{g}}dt, \quad f = \hat{g}^2 - \frac{m}{r^3} + \frac{Q^2}{r^4} \,. \qquad (4.5)
$$

The S^7 metric can be written in the Hopf fibration form,

$$
\sum_{i=1}^{4} \left(d\mu_i^2 + \mu_i^2(d\phi_i + \mathcal{A})^2\right) = (d\psi + \mathcal{P} + \mathcal{A})^2 + ds^2_{\mathbb{CP}^3} \qquad (4.6)
$$

where,

$$
\begin{aligned}
\phi_1 &= \psi + \frac{1}{4}(\xi_1 + \xi_2 + \xi_3), \quad \phi_2 = \psi + \frac{1}{4}(\xi_2 + \xi_3 - 3\xi_1), \\
\phi_3 &= \psi + \frac{1}{4}(\xi_3 + \xi_1 - 3\xi_2), \quad \phi_4 = \psi + \frac{1}{4}(\xi_1 + \xi_2 - 3\xi_3) \,. \qquad (4.7)
\end{aligned}
$$

The advantage of this form of the metric is in the convenience of expressing equal angular momenta in four transverse planes. They essentially translate into momentum along ψ direction. The one form $\mathcal{P}$ is

$$
\mathcal{P} = \frac{1}{4}(d\xi_1 + d\xi_2 + d\xi_3) - \mu_2^2 d\xi_1 - \mu_3^2 d\xi_2 - \mu_4^2 d\xi_3 \,, \qquad (4.8)
$$

and explicit form of the $\mathbb{CP}^3$ metric is

$$
\begin{aligned}
ds^2_{\mathbb{CP}^3} = \; & d\alpha^2 + \sin^2\alpha d\beta^2 + \sin^2\alpha \sin^2\beta d\gamma^2 \\
& + \cos^2\alpha \sin^2\alpha (d\xi_3 + (d\xi_1 - d\xi_3)\cos^2\beta + (d\xi_2 - d\xi_3)\cos^2\gamma \sin^2\beta)^2 \\
& + \sin^2\alpha \sin^2\beta \cos^2\beta (d\xi_1 - d\xi_3 + (-d\xi_2 + d\xi_3)\cos^2\gamma)^2 \\
& + \cos^2\gamma \sin^2\gamma \sin^2\beta \sin^2\alpha (d\xi_2 - d\xi_3)^2 \,. \qquad (4.9)
\end{aligned}
$$

In terms of the Hopf fibration, the metric and 3 form fields for the rotating M2 brane solution are given by,

$$
\begin{aligned}
ds^2 &= -r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2(dx_1^2 + dx_2^2) + 4(d\psi + \mathcal{P} + \mathcal{A})^2 + 4ds^2_{\mathbb{CP}^3} \,, \\
C_3^M &= -r^3 dt \wedge dx_1 \wedge dx_2 + 2Q(d\psi + \mathcal{P}) \wedge dx_1 \wedge dx_2 \,. \qquad (4.10)
\end{aligned}
$$

Here for simplicity we have set $\hat{g} = 1$.

Since T-duality is not a symmetry of M-theory, we can carry out the TsT transformation only after re-expressing this solution in terms of string theory variables. For this purpose, it is useful to write down the metric and three form field in terms of vielbeins, which for this background are,

$$\begin{aligned}
e^0 &= \frac{2r\sqrt{f}}{\sqrt{1-f}}dx^+, \quad e^1 = \frac{r}{\sqrt{1-f}}\left[(1-f)dx^- - (1+f)dx^+\right] , \\
e^2 &= \frac{dr}{r\sqrt{1-f}}, \quad e^3 = d\alpha, \quad e^4 = \sin\alpha d\beta, \quad e^5 = \sin\alpha\sin\beta d\gamma \, , \\
e^6 &= \cos\alpha\sin\alpha(d\xi_3 + (d\xi_1 - d\xi_3)\cos^2\beta + (d\xi_2 - d\xi_3)\cos^2\ \gamma\sin^2\beta) \, , \\
e^7 &= \sin\alpha\sin\beta\cos\beta(d\xi_3 - d\xi_1 + (d\xi_2 - d\xi_3)\cos^2\gamma) \, , \\
e^8 &= \cos\gamma\sin\gamma\sin\beta\sin\alpha(d\xi_3 - d\xi_2), \quad e^9 = 2(d\psi + \mathcal{P} + \mathcal{A}), \quad e^{10} = r dx_2 .
\end{aligned} \tag{4.11}$$

where,

$$x^+ = \frac{1}{2}(t + x_1), \qquad x^- = \frac{1}{2}(t - x_1) \tag{4.12}$$

and,

$$\frac{l^2}{2}d\mathcal{P} = -\omega_{CP3} = -2(e^3 \wedge e^6 + e^4 \wedge e^7 + e^5 \wedge e^8) \, , \tag{4.13}$$

where, ω_{CP3} is the Kähler form on $\mathbb{CP}^3$. We can dimensionally reduce the M-theory solution along any isometry direction and the reduced ten-dimensional geometry is guaranteed to be a solution of the type IIA theory. Consider an M-theory solution of the form,

$$\begin{aligned}
ds^2_M &= ds^2_{10} + e^{2\sigma(x^\mu)}(dx_{10} + A_\mu dx^\mu)^2, \quad x^\mu = 0, 1, , , , 9 \\
C^M_3 &= \frac{1}{3!}C^M_{PQR}dx^P \wedge dx^Q \wedge dx^R \, .
\end{aligned} \tag{4.14}$$

After reducing the solution along x_{10} direction, we get a solution to the type IIA equations of motion, where the metric is expressed in the string frame. The NS-NS sector of the solution looks like,

$$ds^2_{IIA} = e^{\sigma(x^\mu)}ds^2_{10}, \quad \Phi = \frac{3\sigma(x^\mu)}{2}, \quad B_{\mu\nu} = C^M_{\mu\nu 10}, \tag{4.15}$$

whereas the R-R sector takes the following form,

$$C_\mu = A_\mu, \quad C_{\mu\nu\rho} = C^M_{\mu\nu\rho} \, . \tag{4.16}$$

We will now apply TsT transformation on the type IIA solution and uplift the transformed geometry again to M-theory. Thus the transformed background is a new solution

to M-theory. Notice in case of the rotating M2 brane background we can perform the dimensional reduction of M-theory solution, in two different ways. We can either reduce along the worldvolume direction, *i.e.*, x_2 or we can reduce along the transverse direction ψ. These two procedures give rise to two different solutions of type IIA string theory. While in the first reduction we end up with a fundamental string solution of type IIA, the latter one corresponds to D2 brane solution and therefore the corresponding type IIA background has nontrivial R-R sector field. After getting the TST transformed type IIA solution, the uplifted M-theory solution, in general, takes the following form,

$$\begin{aligned} d\hat{s}_M^2 &= e^{-\frac{2\hat{\Phi}}{3}}(d\hat{s})^2 + e^{\frac{4\hat{\Phi}}{3}}(dx_{10} + \hat{C}_{1\mu}dx^\mu)^2 \\ \hat{C}_3^M &= \frac{1}{3!}\hat{C}_{3\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2}\hat{B}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge dx^{10}. \end{aligned} \tag{4.17}$$

In the next couple of subsections we will look at these two reductions and their TsT transforms in detail.

4.1 Reduction along Worldvolume x_2 Direction

Let us consider that the worldvolume direction x_2 is compact. We can then dimensionally reduce the M2 brane solution to a ten dimensional solution of type IIA theory. This ten dimensional solution will only have the NS-NS sector fields turned on. Now following (4.15), we can read out the metric, dilaton and NS-NS two form field,

$$\begin{aligned} ds_{IIA}^2 &= r\left[-r^2 f(r)dt^2 + \frac{dr^2}{r^2 f(r)} + r^2 dx_1^2 + 4(d\psi + \mathcal{P} + \mathcal{A})^2 + 4ds_{\mathbb{CP}^3}^2\right], \\ B_{\mu\nu} &= C^M_{\mu\nu x_2} = -r^3 dt \wedge dx_1 + 2Q(d\psi + \mathcal{P}) \wedge dx_1\,, \\ \exp[\frac{4}{3}\Phi] &= \exp[2\sigma] = r^2\,. \end{aligned} \tag{4.18}$$

The metric and 2 form fields are given by,

$$ds_{10}^2 = r\left(-(e^0)^2 + (e^1)^2 + (e^2)^2 + 4((e^3)^2 + \cdots + (e^8)^2) + (e^9)^2\right), \tag{4.19}$$

$$\begin{aligned} B &= 16r\left(1 + \frac{Q^2}{4r^4}\right)\frac{e^0 \wedge e^1}{\sqrt{4f}} + \frac{2Q}{r\sqrt{1-f}}(\sqrt{f}e^0 + e^1) \wedge e^9 \\ H &= 24r\sqrt{\frac{1-f}{f}}e^0 \wedge e^1 \wedge e^2 + \frac{4Q}{r\sqrt{1-f}}\omega_{CP3} \wedge (\sqrt{f}e^0 + e^1)\,. \end{aligned} \tag{4.20}$$

We follow the TsT transformation rules given in section 2, (2.8,2.12) and get,

$$\begin{aligned} d\hat{s}^2 &= \mathcal{M}\left[\left(A_1 dx^- + K_1 dx^+\right)^2 + \left(A_2 d\psi + 2\sqrt{r}\mathcal{P} + A_3 dx^- + K_2 dx^+\right)^2\right] \\ &\quad -A_4(dx^+)^2 + 4r ds_{\mathbb{CP}^3} + \frac{dr^2}{rf}\,, \end{aligned} \tag{4.21}$$

$$\hat{B} = \mathcal{M}\Big[B(1-2Q\gamma)$$
$$-\gamma A_1 A_2 \left(A_1 dx^- + K_1 dx^+\right) \wedge \left(A_2 d\psi + 2\sqrt{r}\mathcal{P} + A_3 dx^- + K_2 dx^+\right)\Big], \quad (4.22)$$

where,

$$A_1 = r\sqrt{r(1-f)}, \quad A_2 = 2\sqrt{r}, \quad K_1 = -\frac{r^{3/2}(1+f)}{\sqrt{1-f}}, \quad A_3 = K_2 = 2\mathcal{A}_t\sqrt{r}\,,$$
$$A_4 = \frac{2\sqrt{r^3 f}}{\sqrt{1-f}}, \quad \hat{K}_1 = K_1 + \gamma\frac{4fQr^{3/2}}{\sqrt{1-f}}, \quad \hat{K}_2 = K_2 - \gamma\frac{4(2Q^2+r^4)}{\sqrt{r}}\,,$$
$$\mathcal{M} = (1 - 4\gamma Q + 4\gamma^2(Q^2 + (1-f)r^4))^{-1}\,. \quad (4.23)$$

By carrying out expansion of the metric for large r, it is easy to see that the above solution does not have asymptotic Schrödinger geometry. It is particularly easy to see that $\mathcal{M}$ does not become constant as we approach the boundary, it instead vanishes. Thus, although the above field configuration is a solution of type IIA theory and therefore its uplift along x_2 is a solution in M-theory, it does not possess either asymptotic AdS or Schrödinger symmetry. We will therefore not pursue detailed study of this background any further.

We can nevertheless get interesting M-theory solutions with asymptotic AdS symmetry if we follow a strategy of first performing a chain of duality transformations and then doing the TsT transformation. In the process we will have multiple ways of lifting the solution back to M-theory. Following will be our strategy of obtaining a new solution from the type IIA background that we got by reducing rotating M2-brane solution along x^2 direction:

1. We will first T-dualize the type IIA solution (4.18) along ψ direction. This will take us to type IIB theory. We will continue to refer the T-dual direction as ψ.

2. Next we perform an S-duality transformation on this solution.

3. We will then carry out the usual TsT transformation on this S-dual solution. The T-duality will be performed on ψ and the shift will be performed along one of the lightcone directions as defined in (4.12).

4. We can then go back to type IIA solution, either by

   (a) performing another S-duality transformation on the TsT transformed geometry and then T-dualize it back to type IIA or

   (b) by directly T-dualizing the deformed geometry to type IIA.

5. Finally we uplift the resulting type IIA solution obtained by either method to M-theory.

We provide details of intermediate steps in appendix (B) and present only the final TsT transformed solution uplifted to M-theory here. If we follow the step $4a$, then the M-theory metric becomes,

$$\begin{aligned} d\hat{s}^2_M &= r^2\mathcal{M}\big[(1-f-4f\gamma^2)(dx^+)^2 - 2(1+f)dx^+dx^- + (1-f)(dx^-)^2\big] \\ &\quad + \frac{dr^2}{r^2 f} + 4(d\psi^2 + ds^2_{\mathbb{CP}^3}) + e^{\frac{4}{3}\hat{\Phi}}(dx_2 + C^{T2}_{1\mu}dx^\mu)^2 \,, \\ C^M_3 &= \frac{1}{3!}C^{T2}_{3\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2}B^{T2}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge dx_2 \,, \end{aligned} \tag{4.24}$$

with $\mathcal{M} = (1+\gamma^2(1-f))^{-1}$. Expressions for $C^{T2}_{1\mu}$ and $C^{T2}_{3\mu\nu\rho}$ are given in appendix (B).

In case we directly T-dualize the TsT transformed metric using the step 4 (b), it gives the following M-theory geometry,

$$\begin{aligned} ds^2_M &= (1-f)r^2\left((dx^-)^2 + (dx^+)^2 - 2\frac{1+f}{1-f}dx^-dx^+\right) + \frac{1}{fr^2}dr^2 + \mathcal{M}r^2d\psi^2 \\ &\quad + r^2\gamma((1+f)dx^+ - (1-f)dx^-)d\psi + 4(ds^2_{\mathbb{CP}^3} + (dx_2 + C^{T2}_{1\mu}dx^\mu)^2), \\ C^M_3 &= \frac{1}{3!}C^{T2}_{3\mu\nu\rho}dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{2}B^{T2}_{\mu\nu}dx^\mu \wedge dx^\nu \wedge dx_2. \end{aligned} \tag{4.25}$$

The asymptotic geometry is given by,

$$\begin{aligned} ds^2 &= \frac{dr^2}{r^2} + r^2\left(-4dx^+dx^- + 4\gamma dx^+d\psi + d\psi^2\right) \\ &\quad + 4\left(dx_2 + \sum(-\frac{1}{4} + \mu^2_{i+1})d\xi_i\right)^2 + 4ds^2_{\mathbb{CP}^3} \,. \end{aligned} \tag{4.26}$$

Although the deformed M-theory solution has a non-trivial γ dependence and is well defined at the boundary but it is easy to check that this metric is asymptotically AdS. It does not have a Schrödinger like behavior at the boundary.

4.2 Reduction along Transverse ψ Direction

We can also reduce the M-theory solution along ψ direction, the corresponding type IIA solution has both NS-NS and R-R sector turned on. Again following the general rules given in (4.15), the type IIA solution that we get is as follows,

$$\begin{aligned} ds^2_{IIA} &= 2(-r^2fdt^2 + \frac{dr^2}{r^2f} + r^2(dx_1^2 + dx_2^2) + 4ds^2_{\mathbb{CP}^3}) \,, \\ B_2 &= 2Qdx_1 \wedge dx_2, \qquad e^{2\Phi} = 8 \,, \\ C_\mu &= \mathcal{P} + \mathcal{A}, \qquad C_3 = (-r^3dt + 2Q\mathcal{P}) \wedge dx_1 \wedge dx_2 \,. \end{aligned} \tag{4.27}$$

We will perform TsT transformation on this solution. For this purpose, we will again define light-cone directions as in (4.12). In terms of the vielbeins the metric looks as,

$$ds^2 = 2(-(e^0)^2 + (e^1)^2 + (e^2)^2 + 4((e^3)^2 + \cdots + (e^8)^2) + (e^{10})^2)\,. \tag{4.28}$$

We also redefine ξ_1, ξ_2, ξ_3 in terms of $\chi_1 = \xi_3$, $\chi_2 = \xi_1 - \xi_2$ and $\chi_3 = \xi_1 - \xi_3$. The CP_3 metric can now be written using the new variables as,

$$4ds^2_{\mathbb{CP}^3} = \sin^2 2\alpha (d\chi_1 + N)^2 + ds_5^2, \tag{4.29}$$

where,

$$N = d\chi_2 \cos^2\beta + d\chi_3 \cos^2\delta \sin^2\beta\,, \tag{4.30}$$

and ds_5^2 is the remaining part of CP_3 metric consisting of $\alpha, \beta, \delta, \chi_2, \chi_3$.

Next we perform T-duality transformation along χ_1 and the shift along x^- direction. The final TsT transformed solution takes the following form,

$$\begin{aligned} d\hat{s}^2 =&\ 2\mathcal{M}((e^1)^2 + (e^6 + 2\gamma\frac{Q}{r}\sin 2\alpha\, e^{10})^2) \\ &+ 2(-(e^0)^2 + (e^2)^2 + 4((e^3)^2 + \cdots + (e^5)^2 + (e^7)^2 + (e^8)^2) + (e^{10})^2)\,. \end{aligned} \tag{4.31}$$

In coordinate basis, the full solution looks like,

$$\begin{aligned} d\hat{s}^2 &= \mathcal{M}((A_1 dx^- + K_1 dx^+)^2 + (A_2(d\chi_1 + N) + 2\gamma Q \sin 2\alpha\, dx_2)^2 - D(dx^+)^2 \\ &\quad + 2(\frac{dr^2}{r^2 f} + r^2 dx_2^2 + ds_5^2), \\ \hat{B} &= \mathcal{M}\left(B - \gamma A_1 A_2 (A_1 dx^- + K_1 dx^+) \wedge A_2(d\chi_1 + N) + 16\gamma^2 f Q r^2 \sin^2 2\alpha dx^+ \wedge dx_2\right), \\ e^{2\hat{\Phi}} &= \frac{8}{\mathcal{M}}, \quad \hat{C}_1 = \mathcal{P} + \mathcal{A} + 2\gamma Q \cos\alpha\, dx_2 \\ \hat{C}_3 &= C_3 + (\mathcal{P} + \mathcal{A}) \wedge B - \hat{C}_1 \wedge \hat{B}\,. \end{aligned} \tag{4.32}$$

Various functions appearing in the metric and other background fields are,

$$\begin{aligned} A_1 &= \sqrt{2(1-f)}r, \quad A_2 = \sqrt{2}\sin 2\alpha, \\ K_1 &= -\frac{\sqrt{2}(1+f)r}{\sqrt{1-f}}, \quad A_4 = \frac{2\sqrt{2f}r}{\sqrt{1-f}}\,. \end{aligned} \tag{4.33}$$

The asymptotic metric (after uplifting to 11 dimensions) is given by,

$$\begin{aligned} ds^2 =& \left(-16\gamma^2 \sin^2 2\alpha\ r^4 (dx^+)^2 - 4r^2 dx^+ dx^- + r^2 dx_2^2 + \frac{dr^2}{r^2}\right) \\ &+ \frac{1}{2}(A_2(d\chi_1 + N) + 2\gamma Q \sin 2\alpha\, dx_2)^2 + ds_5^2 + 4(d\psi + \hat{C}_1)^2\,. \end{aligned} \tag{4.34}$$

This metric has an interesting feature, that the g_{++} component of it depends on α, which is one of the $\mathbb{CP}^3$ coordinates. As a result the four dimensional space is non-trivially fibered over the periodic coordinate α. Interestingly the α dependence of g_{++} is such that for $\alpha = 0,\ \pi/2$ and π the asymptotic metric reduces to pure AdS_4 spacetime, whereas for all other values of α we get asymptotically Schrödinger background. We will get back to these special values of α in the concluding section. Defining rescaled light-cone coordinates as,

$$u = 2\nu x^{+}, \qquad v = \frac{1}{\nu}x^{-}, \quad \nu = 2\gamma \sin 2\alpha\,, \tag{4.35}$$

it is easy to see that the above metric takes the form (2.1) at the boundary $r \to \infty$ with dynamical exponent $z = 2$. The fact that the asymptotic geometry is nontrivially fibered over a compact coordinate of $\mathbb{CP}^3$ has interesting implications for thermodynamics. We will find it convenient to isolate the coordinate α from rest of the $\mathbb{CP}^3$ coordinates and club it with the noncompact coordinates in order to give unambiguous definitions of thermodynamic quantities. In particular, we will leave out the α direction and integrate out rest of the $\mathbb{CP}^3$ coordinates along with the Hopf fiber coordinate ψ. This will leave us with a five dimensional spacetime, which as mentioned above looks asymptotically like a four dimensional Schrödinger space fibered over α circle. All the thermodynamic quantities associated with the black hole in the interior of this spacetime will depend on where we are located in α direction.

5. Thermodynamics and Phase Structure

We will study thermodynamics and possible phase transition of these non relativistic systems to other non-relativistic systems. To understand the phase structure of a black hole spacetime one needs to compute its free energy[36] which is given by

$$W = \frac{I}{\beta}\,, \tag{5.1}$$

where, I is the (Euclidean) on-shell action and β is the inverse temperature. In asymptotically AdS space the on-shell action has large volume divergences[8] and we need to regularize this action by either using boundary counterterms [37] or subtracting contribution of background spacetime [38]. A detailed discussion of background subtraction method can be found in [39].

[8]For asymptotically flat black holes the divergence comes from the Gibbons-Hawking boundary term.

In the Hawking-Page formalism (or background subtraction method), one has to match the geometry of black hole spacetime and background spacetime at some constant $r = \tilde{R}$ hyper-surface. This matching fixes the temperature of background spacetime in terms of that of the black hole spacetime, and at the end one takes the cutoff $\tilde{R} \to \infty$. In [12, 13] it has been argued that for black holes in asymptotically Schrödinger space, the subtraction method is subtle as the null circle becomes degenerate for background spacetime on constant $\tilde{R}$ hyper-surface, whereas for black hole spacetime it is not. Therefore they introduced an "*ad hoc*" prescription to compute renormalized on-shell action.

It is worth emphasizing that the Euclidean method, by itself, is sufficient to compute all thermodynamic quantities, which satisfy the first law, starting from the renormalized on-shell action. The on-shell action (or more precisely the free energy) captures phase structure of the black hole spacetime. Therefore, our goal here is to compute the free energy so that we can analyze the phase structure of these solutions. While the Euclidean method is sufficient, it can become quite cumbersome for complicated field configurations. Our new solutions with Schrödinger asymptotics do indeed have several fields turned on. While it is still possible to evaluate the Euclidean action, we will momentarily define an alternate method for computing the on-shell Euclidean action. First of all notice that the Hawking temperature can be determined either by computing surface gravity on the horizon or by computing periodicity of the Euclidean time circle. In either case, it suffices to have knowledge of the near horizon geometry. Secondly, recall entropy of black hole spacetime can be determined in two different ways. One way is to evaluate the Euclidean on-shell action in the black hole spacetime and then using thermodynamic relations to derive expression for entropy. Another method is to use Wald's formula to directly derive black hole entropy. In all known relativistic examples, it is known that both these methods give rise to same formula for entropy[39]. The strategy we will employ rests on the equality of these two ways of deriving the entropy. We will first determine the entropy using Wald's method and turn the Euclidean action method on its head to derive the on-shell action by integrating the Wald entropy. This requires solving a first order inhomogeneous differential equation. The on-shell action derived in this manner contains an ambiguity corresponding to the constant of integration. This ambiguity can be fixed by demanding that the free energy vanishes for a fixed background spacetime. We will subsequently write down the free energy for the black hole spacetime using this on-shell action and the Hawking temperature derived from either of the methods mentioned above. We will illustrate this method for the AdS Reissner-Nordström black hole in d+1 dimensions, however, we will eventually apply it to the asymptotically Schrödinger backgrounds. This method becomes particularly efficient in case of the TsT transformed geometers where a variety

of fields are turned on due to solution generating technique. While the Euclidean action can nevertheless be determined by conventional methods, our method turns out to be much more efficient in deriving desired results.

Let us start with AdS-Reissner-Nordström black hole in d+1 dimensions. The solution is given by,

$$\begin{aligned} ds^2 &= -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2 d\Omega_{k,d-1}^2\,, \\ A(r) &= \left(-\frac{1}{c}\frac{q}{r^{d-2}} + \mu\right) dt\,, \end{aligned} \tag{5.2}$$

where,

$$V(r) = k - \frac{m}{r^{d-2}} + \frac{q^2}{r^{2d-4}} + \frac{r^2}{L^2}, \qquad c = \sqrt{\frac{2(d-2)}{d-1}}. \tag{5.3}$$

Here k determines the horizon topology. For flat horizon (black brane), $k = 0$, whereas $k = 1(-1)$ for spherical (hyperboloid) horizons. The asymptotic value of the gauge field A_t is defined to be the chemical potential,

$$\mu = \frac{1}{c}\frac{q}{r^{d-2}}. \tag{5.4}$$

The temperature of the black hole can be computed from its surface gravity at the horizon defined as [8],

$$\kappa^2 = -\frac{1}{2}(\nabla^a \zeta^b)(\nabla_a \zeta_b)|_H, \tag{5.5}$$

where ζ^a is the null generator of the horizon. The Hawking temperature is then, $T = \frac{1}{2\pi}\kappa$. We can also find the temperature as the inverse of the period of the Euclidean time circle. Thus the classical solution has a finite temperature

$$T = \frac{1}{\beta} = \left[\frac{4\pi L^2 r_+^{2d-3}}{d r_+^{2d-2} + k(d-2)L^2 r_+^{2d-4} - (d-2)q^2 L^2}\right]^{-1}, \tag{5.6}$$

where r_+ is the position of the outer horizon.

We compute entropy using Wald's formula. For two derivative gravity theory, the formula gives entropy as the horizon area $\mathcal{A}_{\text{Horizon}}$. To employ Wald's formula, we need the metric to be written in Einstein's frame. We find the horizon area [40] by writing the metric in the Boyer-Lindquist coordinate, so that there is a r-coordinate which does not mix with others[9]. We can then write the metric as

$$g_{\mu\nu} = \left(\begin{array}{c|c} g_{rr} & 0 \\ \hline & \\ 0 & P_{ij} \\ & \end{array}\right). \tag{5.7}$$

[9] All the metrics that we have considered in this paper are in the Boyer-Lindquist coordinates.

Note that the time coordinate t appears in the metric P_{ij}. The local definition of the horizon is a fixed (t, r) surface where g_{rr} diverges and the determinant of P_{ij} vanishes. The cofactor of P_{ij} is,

$$A^{\mu\nu} = g^{\mu\nu} detP \,. \tag{5.8}$$

The area of the horizon can then be written as $\mathcal{A}_{\text{Horizon}} = \sqrt{A^{tt}}$. Using this formulation, we get entropy

$$S = \frac{\mathcal{A}_{\text{Horizon}}}{4G} = \frac{V_{k,d-1} r_+^{d-1}}{4G}, \tag{5.9}$$

where, $V_{k,d-1}$ is the volume of the transverse space. In the next subsection we will use thermodynamic relations to derive the on-shell action using the expression for entropy given in (5.9).

5.1 Computation of On-Shell Action

We will now use the expressions derived above for the entropy and the temperature and substitute them in thermodynamics relations to compute the on-shell action or equivalently the free energy. The expression for free energy depends on the ensemble we are working on. In the present situation we have two ensembles: fixed potential ensemble and fixed charge ensemble[41, 42].

Fixed Potential case

In this ensemble the relation between entropy and free energy is given by,

$$S - \beta \left(\frac{\partial I}{\partial \beta}\right)_{\mu} + I = 0 \tag{5.10}$$

Once we determine the on-shell action I we can compute other thermodynamic quantities, like the energy and the physical charge using

$$\begin{aligned} E &= \left(\frac{\partial I}{\partial \beta}\right)_{\mu} - \frac{\mu}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_{\beta} , \\ Q &= -\frac{1}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_{\beta} . \end{aligned} \tag{5.11}$$

Now let us solve equation (5.10) using (5.6) and (5.9). The solution is given by,

$$I = -\frac{V_{k,d-1}\beta}{16\pi G L^2}\left(L^2 r_+^{d-2}(c^2\mu^2 - k) + (r_+^n - 4c_1)\right). \tag{5.12}$$

However as we pointed earlier, we have an undetermined constant c_1, which has to be fixed. This is achieved by employing the following strategy: the free energy of the black hole space-time is measured with respect to "some" background space-time. We, therefore, choose the scale such that the free energy of the background is zero. With this choice, we can fix the constant c_1. If we want to measure the free energy of the black hole spacetime with respect to global AdS then we set the free energy of the global AdS spacetime to be *zero*. The global AdS spacetime corresponds to $r_+ = 0$ keeping μ fix in this ensemble[10]. This implies

$$W = \frac{I}{\beta}\bigg|_{r_+ \to 0} = 0. \tag{5.13}$$

This uniquely fixes $c_1 = 0$. Therefore the on-shell action is given by,

$$I = -\frac{V_{k,d-1}\beta}{16\pi G L^2}\left(L^2 r_+^{d-2}(c^2\mu^2 - k) + r_+^n\right). \tag{5.14}$$

This result was obtained in [41] and the complete phase structure of the AdS-Reissner-Nordström black hole was analyzed there[11].

Fixed Charge case

One can also consider thermodynamic ensemble where the charge parameter q is fixed. In this case the thermodynamic relations are given by,

$$S - \beta\left(\frac{\partial I}{\partial \beta}\right)_q + I = 0\,, \tag{5.15}$$

and

$$\begin{aligned} E &= \left(\frac{\partial I}{\partial \beta}\right)_q, \\ \mu &= \frac{1}{\beta}\left(\frac{\partial I}{\partial \mu}\right)_\beta. \end{aligned} \tag{5.16}$$

We use (5.15) to compute I and it is given by,

$$I = \frac{V_{k,d-1}\beta}{16\pi G L^2}\left(kL^2 r_+^{d-2} - r_+^d + (2d-3)q^2 L^2 r_+^{2-d} + 4c_1\right). \tag{5.17}$$

[10] For black-branes, we chose the background to be AdS soliton and set the free energy of the soliton to zero.

[11] Also see [43] for black brane phase structure.

For this ensemble, we can not measure the free energy of the black hole spacetime with respect to global AdS as the later has zero charge. We will instead measure the free energy with respect to the extremal solution. The extremal solution has same charge q as the black hole, whose free energy we wish to derive and its horizon is located at r_e where,

$$d\, r_e^{2+2d} + L^2(d-2)(q^2 r_e^4 - k r_e^2 d) = 0. \tag{5.18}$$

In this case we are comparing the free energy of the black hole with that of the extremal solution. The constant of integration, c_1, is thus determined by setting the free energy for the extremal black hole to zero.

$$W = \frac{I}{\beta}\bigg|_{r_+ \to r_e, q \text{ fix}} = 0 \tag{5.19}$$

gives,

$$c_1 = \frac{1}{4} r_e^{-d-2} \left(r_e^{2d+2} - L^2((2d-3)q^2 r_e^4 + k r_e^{2d}) \right). \tag{5.20}$$

The Euclidean action for the black hole is therefore given by,

$$I = \frac{V_{k,d-1}\beta}{16\pi G L^2} \left(k L^2 r_+^{d-2} - r_+^d + (2d-3) q^2 L^2 r_+^{2-d} - \frac{2k(d-1)}{d} L^2 r_e^{d-2} - \frac{2(d-1)^2}{d} \frac{q^2 L^2}{r_e^{d-2}} \right). \tag{5.21}$$

This result was also found in [41]. For $k = 0$ (black brane) case, we see that the black brane geometry is always dominant over the background geometry. Thus we see that our strategy gives the desired result in case of AdS-Reissner-Nordström black hole in d+1 dimensions. In the next subsection we will use this strategy to study thermodynamics of non-relativistic black objects.

5.2 Thermodynamics of Non-Relativistic Black Objects

We will use our strategy to do similar computations but this time for $(d+1)$ dimensional non-relativistic black objects (black branes and black holes) and study their phase structure. Such black objects can be obtained by dimensional reduction of various non-relativistic branes (D3, M2) along the transverse space. As we have seen in the last subsection, for studying the phase structure of a black hole space-time, we need to know its free energy. Using thermodynamic relations, we can obtain the free energy from the knowledge of the temperature and the entropy of the black hole.

To find the temperature of the non-relativistic black objects, we can use the surface gravity defined in (5.5). The null generator ζ is proportional to $\frac{\partial}{\partial t}$. In analogy with asymptotically flat space, where the null generator has unit norm asymptotically, we

can fix the normalization of ζ . We demand that the component of ζ along the boundary (non-relativistic CFT) time-translation has unit coefficient. Since the generator of time translation in the boundary theory is $\frac{\partial}{\partial u}$ we get

$$\zeta = \frac{1}{\nu}\frac{\partial}{\partial t} = \frac{\partial}{\partial u} + \frac{1}{2\nu^2}\frac{\partial}{\partial v}\,, \tag{5.22}$$

where, ν is the scaling parameter of the light-cone directions, which is required for obtaining the boundary $r \to \infty$ metric as in (2.1). In particular, $\nu = \gamma l$ for non-relativistic rotating D3 brane geometry (3.15) and $\nu = 2\gamma \sin 2\alpha$ for non-relativistic rotating M2 brane geometry (4.35). We can also find the temperature as the inverse of the period of the Euclidean time circle. Either of these procedures give us,

$$T = \frac{1}{\beta} = \left[\frac{4\pi r_+^{2d-3}\nu}{d r_+^{2d-2} + k(d-2) r_+^{2d-4} - (d-2)q^2}\right]^{-1}. \tag{5.23}$$

To compute the entropy, we need to compute the horizon area $\mathcal{A}_{\text{Horizon}}$. We can use (5.7) and (5.8), provided we keep in mind that the correct time direction for the non-relativistic black objects is u. This gives us the entropy as,

$$S = \frac{\mathcal{A}_{\text{Horizon}}}{4G} = \frac{V_{k,d-1} r_+^{d-1}\nu}{4G}. \tag{5.24}$$

5.2.1 On-Shell Action for Fixed Potential Case

To study the phase structure, we have to first decide on the choice of ensemble. Now that the non-relativistic theories can be realized as a deformed version of relativistic quantum field theories, it is natural to allow the particle number to fluctuate. Thus grand canonical ensemble is the right choice of ensemble in this case. Hence we study the phase structure of non-relativistic systems only for the fixed potential ensemble. The strategy described above can be readily applied to the fixed charge case, however, we will not pursue that line here. We hope to report on that elsewhere.

To compute the on-shell action, we use the temperature and entropy as input and use relation (5.10). In this case, we have two chemical potentials corresponding to two conserved charges. As the killing generator ζ of the event horizon has a component along the light-like direction v (5.22), the corresponding momentum $\frac{\partial}{\partial v}$ is a conserved charge and we have a chemical potential for v-translation given as,

$$\mu_1 = \frac{1}{2\nu^2}. \tag{5.25}$$

The other chemical potential is associated with the boundary value of the gauge field (5.4). With u being the correct boundary time direction, we decompose $A = A_t dt = A_u du + A_v dv$. Thus the second chemical potential is defined as,

$$\mu_2 = \lim_{r\to\infty} A_u = \frac{1}{2\nu c}\frac{q}{r_+^{d-2}} = \frac{\sqrt{\mu_1}}{\sqrt{2}c}\frac{q}{r_+^{d-2}}\,. \tag{5.26}$$

Replacing the charges in-terms of the corresponding chemical potentials, the temperature in grand canonical ensemble is given by,

$$\frac{1}{T} = \beta = \frac{2\sqrt{2}\pi r_+\sqrt{\mu_1}}{-2c^2 d\mu_2^2 + 4c^2\mu_2^2 + (d-2)k\mu_1 + dr_+^2\mu_1}. \tag{5.27}$$

We can then solve the eq.(5.10) to write the on-shell action as,

$$I = \frac{-2\sqrt{2}c^2\mu_2^2 r_+^d + 8c_1 r_+^2\sqrt{\mu_1} + \sqrt{2}k\mu_1 r_+^d - \sqrt{2}\mu_1 r_+^{d+2}}{8r_+\sqrt{\mu_1}\left(-2c^2 d\mu_2^2 + 4c^2\mu_2^2 + (d-2)k\mu_1 + dr_+^2\mu_1\right)}\,. \tag{5.28}$$

The corresponding free energy is given by,

$$W = \frac{r_+^d\left(k\mu_1 - 2c^2\mu_2^2\right) + 4\sqrt{2}c_1 r_+^2\sqrt{\mu_1} - \mu_1 r_+^{d+2}}{16\pi r_+^2\mu_1}\,. \tag{5.29}$$

As in the relativistic case, we have an undetermined constant c_1 that needs to be fixed. Choosing the non-relativistic extremal black hole/brane geometry[12] as the background and setting its free energy to zero, we get $c_1 \to 0$ and the free energy becomes,

$$W = \frac{r_+^d\left(k\mu_1 - 2c^2\mu_2^2\right) - \mu_1 r_+^{d+2}}{16\pi r_+^2\mu_1}. \tag{5.30}$$

Thus, from (5.30) we see that black branes (k=0) are always dominant[13] whereas, there exists a phase transition between the black hole (k=1) phase and the background spacetime when

$$\frac{\mu_1}{\mu_2^2} = \frac{2c^2}{1-r_+^2}. \tag{5.31}$$

When the ratio $\frac{\mu_1}{\mu_2^2}$ is less than its critical value (5.31) the black hole geometry will be dominant over the background. This is the usual Hawking-Page transition. Also, if we assume that both the chemical potentials are positive, then $r_+ < 1$.

It is clear from the equation of β (5.27), that there are two distinct behaviour of β as a function of the ratio of two chemical potentials, namely μ_1/μ_2^2. For $\frac{\mu_1}{\mu_2^2} < 2c^2$, β

[12] This geometry is obtained by setting $r_+ \to 0$ and $Q \to 0$ of the higher dimensional brane solution.

[13] The black brane can have a transition to the AdS soliton geometry[26, 44]

diverges when[14] $r_+ = \frac{(d-2)(2c^2\mu_2^2-\mu_1)}{d\mu_1}$, whereas it smoothly goes to zero as $r_+ \to 0$ for $\frac{\mu_1}{\mu_2^2} > 2c^2$. In the first case there exists a unique black hole associated with each temperature and this branch dominates the thermodynamics (free energy is always negative). In the second case, for a fixed chemical potentials there exists a nucleation temperature

$$\frac{1}{T_n} = \beta_n = \sqrt{\frac{2\pi^2}{d(d-2)(\mu_1 - 2c^2\mu_2^2)}}, \tag{5.32}$$

at which two black holes with same horizon radii are formed with $r_n = \sqrt{\frac{(d-2)(\mu_1-2c^2\mu_2^2)}{d\mu_1}}$. As we increase the temperature one of them becomes smaller (small black hole) and the other one becomes larger (big black hole). For temperatures greater that T_N, these two black holes have horizon radius $r_+ = \lambda r_n$ where, $\lambda > 1$ for big black hole and $\lambda < 1$ for small black hole. We compute free energies for these two black holes and it turns out that free energy for small black hole is always positive and therefore this phase is unstable whereas the big black hole phase is dominant over background spacetime for $\lambda > \sqrt{\frac{d}{d-2}}$ which is compatible with (5.31). The Hawking-Page transition temperature is,

$$T_{HP} = (d-1)\sqrt{\frac{\mu_1 - 2c^2\mu_2^2}{2\pi^2}}. \tag{5.33}$$

We can also find conserved charges corresponding to the chemical potentials. They are given by[15],

$$\begin{aligned} P_1 &= -\frac{\partial W}{\partial \mu_1}\bigg|_{T,\mu_2} = -\frac{\nu^2 r_+^{-d-2}\left(d\left(r_+^{2d}\left(k+r_+^2\right)+q^2r_+^4\right)-2kr_+^{2d}\right)}{16\pi}, \\ P_2 &= -\frac{\partial W}{\partial \mu_2}\bigg|_{T,\mu_1} = \frac{\sqrt{2(d-2)(d-1)}q\nu}{4\pi}. \end{aligned} \tag{5.35}$$

These are the physical charges for generic $(d+1)$ dimensional non-relativistic black objects. In particular for 5 dimensional charged non-relativistic black branes, above results match exactly with those of [26][16].

The charge P_2 obtained for the non-relativistic case is 2ν times that of the relativistic charge of [41]. This is easily understood from the fact that the gauge field here is A_u.

[14] we have considered $d > 2$.

[15] We have rewrite r_+ in-terms of temperature as

$$r_+ = \frac{\sqrt{2}\pi T\sqrt{\mu_1} - \sqrt{\mu_1\left(2\pi^2T^2-(d-2)d\left(k\mu_1-2c^2\mu_2^2\right)\right)}}{d\mu_1}, \tag{5.34}$$

we choose this branch as $T \to \infty$ when $r_+ \to 0$.

[16] Please note that the chemical potential μ_2 is $\frac{1}{c}$ times their μ_2.

6. Discussion

We analyzed TsT transformed geometries associated with brane configurations in type II string theory as well as in M-theory. One of these configurations was studied earlier in the literature. We find that the space-like TsT transformation in the rotating D3 brane case commutes with the extremal near horizon limit. The geometry obtained by TsT transforming the $D1$-$D5$-p solution has the feature that in the extremal near horizon limit it reduces to the undeformed extremal near horizon geometry. The reason is that in this limit γ dependence drops out completely, which essentially erases the memory of the shift transformation. In case of the BTZ black holes, the dual field theory is non-relativistic conformal quantum mechanics.

The TsT transformation of M2 brane geometry is carried out by using a set of U-duality transformations. We carry out these transformations by first reducing the system to type II set up and then doing TsT and lift it back to the M-theory solution. Among various ways of generating new M-theory solution, we find that the null TsT transformation on the type IIA background obtained by reduction along ψ direction has the feature that the metric is singular if one of the compact coordinate $\alpha = 0,\ \pi/2,\ \pi$. Asymptotically this metric approaches Schrödinger metric for generic α but for these special values of $\alpha = 0,\ \pi/2,\ \pi$, the space becomes AdS. However, for precisely these values of α the metric has curvature singularity. One way to get around this is to consider blow ups of $\mathbb{CP}^3$ at these points so that the singularity is removed. These spaces are $\mathbb{CP}^3$ analogs of del Pezzo spaces and the M-theory solution is well behaved on such blown up spaces.

We also analyzed thermodynamics of these geometries. We have taken a novel approach to derive free energy of the system and our results match with known results in the literature both for fixed charge and fixed chemical potential ensemble. Using the expression for free energy thus derived we have analyzed phase structure of generic non-relativistic $d+1$-dimensional black objects (holes and branes). Since these non-relativistic systems are derived from the relativistic ones, it is natural to choose the ensemble which allows changing particle number, thus the grand canonical ensemble is a natural choice in this case. We have shown that for a specific range of chemical potentials (5.31), there exist a Hawking-Page transition from the non-relativistic black hole to non-relativistic extremal brane geometries. This thermodynamic analysis can be applied to $d+1$ dimensional systems for $d > 2$ only. In particular, it cannot be used for the BTZ black hole case. The entropy and temperature computed for the BTZ background are real quantities, however, if we compute them from the action both of them turn out to be imaginary. While it can be shown that the temperature is analytic, the entropy is not. As a result, knowledge of real temperature and entropy is

not sufficient to deduce the action and hence the free energy uniquely by inverting the thermodynamic relation. This is related to the fact that the Euclidean time circle has imaginary periodicity.

It would be interesting to understand relation of transformed BTZ solution to non-relativistic conformal quantum mechanics better. In particular, extension of quantum entropy function to these cases would be quite illuminating. The 1+1 dimensional non-relativistic conformal field theory obtained from M2 branes may be relevant to the physics of the Burgers equation which is non-relativistic Navier-Stokes equation in one spatial dimension. We hope to address these issues in future.

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Appendix

A. T-duality Transformation Rules for NS-NS and R-R Fields

For self-consistency of this article, in this appendix, we will jot down the T-duality transformation rules for both NS-NS and R-R fields. For details, the reader can referred to original articles [32, 45] and [46]. We will however closely follow the notation of [31].

NS-NS fields:

Let us denote, the T-dualized direction as ψ and other directions are denoted by $a, b, ...$ Transformation of the NS-NS fields, *i.e.*, the metric, the NS-NS two forms B and the dilaton under T-duality is as follows,

$$\begin{aligned} g'_{\psi\psi} &= \frac{1}{g_{\psi\psi}}, \qquad g'_{a\psi} = \frac{B_{a\psi}}{g_{\psi\psi}}, \qquad g'_{ab} = g_{ab} - \frac{g_{a\psi}g_{\psi b} + B_{a\psi}B_{\psi b}}{g_{\psi\psi}} \\ B'_{a\psi} &= \frac{g_{a\psi}}{g_{\psi\psi}}, \quad B'_{ab} = B_{ab} - \frac{g_{a\psi}B_{\psi b} + B_{a\psi}g_{\psi b}}{g_{\psi\psi}}, \quad \Phi' = \Phi - \frac{1}{2}\ln g_{\psi\psi}. \end{aligned} \tag{A.1}$$

R-R fields:

For stating the transformations of different R-R field, we first define some notation. First, we decompose a p-form $\mathcal{N}_p$ as,

$$\mathcal{N}_p = \bar{\mathcal{N}}_p + \mathcal{N}_{p[\psi]} \wedge d\psi, \tag{A.2}$$

where, $\bar{\mathcal{N}}_p$ does not contain any ψ component and $\mathcal{N}_{p[\psi]}$ is a $(p-1)$ formed defined as,

$$(\mathcal{N}_{p[\psi]})_{a_1....a_{p-1}} = (\mathcal{N}_p)_{a_1.....a_{p-1}\psi}. \tag{A.3}$$

Next step is to define one forms,

$$J = \frac{g_{a\psi}}{g_{\psi\psi}} dx^a, \qquad b = B_{[\psi]} + d\psi\,. \tag{A.4}$$

With these definitions, we are ready to state the T-duality transformation rule for the R-R fields. They are,

$$C'_p = C_{p+1[\psi]} + \bar{C}_{p-1} \wedge b + C_{p-1[\psi]} \wedge b \wedge J, \tag{A.5}$$

and similar transformations for the R-R field strengths, $\mathcal{F}_p$.

B. TsT Transformed Metrics

We have laid down the strategy that we follow in the applying TsT transformation to M-theory solution in subsection (4.1). Here we will give intermediate steps of those set of duality transformations on rotating M2-brane solution.

The solution after step 1 is,

$$\begin{aligned} ds^2_{T1} &= r\left[-r^2 f(r) dt^2 + \frac{dr^2}{r^2 f(r)} + 4ds^2_{\mathbb{CP}^3}\right] + r^3(1+\frac{Q^2}{r^4})dx_1^2 + \frac{1}{4r}d\psi^2\,, \\ B^{T1} &= -(r^3 + 2\mathcal{A}_t Q)dt\wedge dx_1 + \mathcal{A}_t dt \wedge d\psi + \sum_{i=1}^{3}(\frac{1}{4} - \mu^2_{i+1})d\xi_i \wedge d\psi\,, \\ \Phi^{T1} &= \Phi - \frac{1}{2}\ln 4r = \ln\frac{r}{2}\,, \quad C_0^{T1} = C_2^{T1} = C_4^{T1} = 0\,. \end{aligned} \tag{B.1}$$

This is a solution in type IIB theory. The string coupling is still divergent in the large r limit. We will use the S-duality symmetry of type IIB theory to transform this solution to a dual frame. The S-duality transformation leaves the metric in Einstein frame and the self-dual four form field invariant and transforms the dilaton-axion as well as the NS-NS and R-R two form fields. Note that the string frame metric does transform under S-duality,

$$\begin{aligned} &ds^2_{S1} = e^{-\Phi^{T1}} ds^2_{T1} \quad e^{\Phi^{S1}} = e^{-\Phi^{T1}} = \frac{2}{r} \\ &B^{S1} = C_2^{T1} = 0 \quad C_2^{S1} = -B^{T1} \quad C_4^{S1} = C_4^{T1} = 0 \end{aligned} \tag{B.2}$$

In the S-dual frame the string coupling becomes weak in the large r limit. We then apply the TsT transformation on the above solution after writing it in terms of the light cone coordinates defined in (4.12). Using (2.4), we get the result,

$$
\begin{aligned}
d\hat{s}^2 &= \mathcal{M}(A_1 dx^- + K_1 dx^+)^2 + \mathcal{M}(A_2 d\psi + A_5 dx^- + A_6 dx^+)^2 - A_4 (dx^+)^2 + 2(4ds^2_{\mathbb{CP}^3} + \frac{dr^2}{r^2 f}) \\
\hat{B} &= -\gamma A_1 A_2 \mathcal{M}(A_1 dx^- + K_1 dx^+) \wedge (A_2 d\psi + A_5 dx^- + A_6 dx^+) \\
e^{2\hat{\Phi}} &= e^{2\Phi^{S1}} \mathcal{M} \quad \hat{C}_2 = C_2^{S1}, \quad \hat{C}_0 = \gamma \mathcal{A}_t, \quad \mathcal{M} = (1+\gamma^2(1-f))^{-1},
\end{aligned} \tag{B.3}
$$

where remaining R-R sector fields vanish, and

$$
\begin{aligned}
A_1 &= r\sqrt{2(1-f)} \quad K_1 = -\frac{\sqrt{2}r(1+f)}{\sqrt{1-f}} \\
A_2 &= \frac{1}{\sqrt{2}r} \quad A_5 = -A_6 = \frac{\sqrt{2}Q}{r} \quad A_4 = \frac{8fr^2}{1-f}
\end{aligned} \tag{B.4}
$$

We then follow step 4 and 5 to get the TsT transformed solution in type IIA theory and finally to M-theory. As mentioned in our strategy, there are two way to arrive at the solution in the type IIA frame. The solution after step 4(a) is,

$$
\begin{aligned}
ds^2_{T2} &= e^{\frac{2}{3}\hat{\Phi}} \big[r^2 \mathcal{M}((1-f-4f\gamma^2)(dx^+)^2 - 2(1+f)dx^+ dx^- + (1-f)(dx^-)^2) \\
&\quad + \frac{dr^2}{r^2 f} + 4(d\psi^2 + ds^2_{\mathbb{CP}^3}) \big] \\
B^{T2} &= -2Q(dx^+ + dx^-) \wedge d\psi, \\
C_1^{T2} &= C^{S2}_{2[\psi]} + C_0 \wedge \tilde{b} = \gamma \mathcal{M} A_1 A_2^2 (A_1 dx^- + K1 dx^+) + \gamma \mathcal{A}_t \tilde{b} \\
C_3^{T2} &= -2(r^3 + 2\mathcal{A}_t Q) dx^+ \wedge dx^- \wedge \bar{b} + \hat{C}_{2[\psi]} \wedge \tilde{b} \wedge \tilde{J}, \qquad C_5^{T2} = 0
\end{aligned} \tag{B.5}
$$

where,

$$
\tilde{b} = -\mathcal{A}_t(dx^+ + dx^-) - \sum_{i=1}^{3} (\frac{1}{4} - \mu^2_{i+1}) d\xi_i + d\psi, \quad \tilde{J} = \frac{A_5 dx^- + A_6 dx^+}{A_2}. \tag{B.6}
$$

On the other hand, if we directly T-dualize the TsT transformed metric, as stated in

step 4(b), it gives the following geometry,

$$\begin{aligned}
ds^2_{T2} &= 2(1-f)r^2\left((dx^-)^2+(dx^+)^2-2\frac{1+f}{1-f}dx^-dx^+\right)+2\mathcal{M}^{-1}r^2d\psi^2 \\
&\quad +2r^2\gamma((1+f)dx^+-(1-f)dx^-)d\psi+\frac{2}{fr^2}dr^2+8ds^2_{\mathbb{CP}^3}, \\
\Phi_{T2} &= \frac{3}{2}\ln 2, \quad B^{T2}=2Q(dx^--dx^+)\wedge d\psi, \\
C_1^{T2} &= \hat{C}_{2[\psi]}+\hat{C}_0\wedge b=-\mathcal{A}_t(dx^++dx^-)-\sum_{i=1}^{3}(\frac{1}{4}-\mu_{i+1}^2)d\xi_i+\gamma\mathcal{A}_t b, \\
C_3^{T2} &= -2(r^3+2\mathcal{A}_tQ)dx^+\wedge dx^-\ \wedge b+\hat{C}_{2[\psi]}\wedge b\wedge J,
\end{aligned} \tag{B.7}$$

where,

$$b=-\gamma A_1A_2^2\mathcal{M}(A_1dx^-+K_1dx^+)+d\psi, \quad J=\frac{A_5dx^-+A_6dx^+}{A_2}. \tag{B.8}$$

Finally, we uplift these solutions to M-theory. The corresponding M-theory geometry is given in section (4).

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